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On Quantum Phase Transition. I. Spinless Electrons Strongly Correlated with Ions

Alain Messager[1]

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We study a large class **F** *of models of the quantum statistical mechanics* dealing with two types of particles. First the spinless electrons are quantum particles obeying to the Fermi statistics, they can hop. Secondly the ions which cannot move, are classical particles. The Falicov–Kimball (FK) model[(1)] is a well known model belonging to **F**, for which the existence of an antiferomagnetic phase transition was proven in the seminal paper of Kennedy and Lieb.[(2)] This result was extended by Lebowitz and Macris.[(3)] A new approach to this problem based on quantum selection of the ground states was proposed in ref. 4. In this paper we extend this approach to show that, under the "strong insulating condition," any hamiltonian of the class **F** admits, at every temperature, an effective hamiltonian, which governs the behaviour of the ions interacting through forces mediated by the electrons. The effective hamiltonians are long range many body Ising hamiltonians, which can be computed by a cluster expansion expressed in term of the quantum fluctuations. *Our main result is that we can apply the powerfull results of the classical statistical mechanics to our quantum models.* In particular we can use the classical Pirogov–Sinai theory to establish a hierarchy of phase diagrams, we can also study of the behaviour of the quantum interfaces,[(29)] and so on…

KEY WORDS: Itinerant electrons; phase transitions; Pirogov–Sinai theory; commensurate phases.

0. INTRODUCTION AND RESULTS

One major problem of the quantum statistical mechanics is that, in general, the hamiltonians cannot be diagonalized. Usually the hamiltonian splits into a classical part (more generally a diagonal part w.r.t. a basis) and a

[1] Centre de Physique Théorique, CNRS - Luminy, Case 907, F-13288 Marseille Cedex 9, France; e-mail: messager@cpt.univ-mrs.fr

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quantum part (off diagonal part). We can distinguish, at least, three typical cases.

Case I. The classical hamiltonian has a finite number of ground states and the contours separating the ground states satisfy the Peierls condition. Then, when the quantum part of the hamiltonian is small, the phase transitions are generally driven by the classical part. One typical example is the existence of a spontaneous magnetization for any anisotropic Heisenberg model in dimension larger than 1.[37] Then we will speak of **classical phase transition**.

Case II. The classical hamiltonian has *either an infinity of ground states or even a finite number, but the Peierls condition is not satisfied.* This is of course one very interesting situation. In many cases the quantum fluctuations will select a finite set of classical ground states, which will generate the coexisting phases at low temperature. This is the case of the Falicov–Kimball model as shown by Kennedy and Lieb in ref. 2. We notice that in this case the ground states of the full hamiltonian are not infinitely degenerated. Then we speak of **quantum phase transition**.

Case III. The classical hamiltonian has an infinity of ground states, and the full hamiltonian has also an infinity of ground states. In some cases the thermal fluctuations will select a finite set of classical ground states, which will generate the phases coexisting at low temperature. Such an example is the 111 interface of the anisotropic Heisenberg model.[38] Then we will also speak of **quantum phase transition**.

The aim of this paper is to study a large class **F** of quantum lattice models containing the Falicov–Kimball model, which modelize the spinless fermion, which are quantum particles, interacting with ions, which are classical particles. More precisely, the electrons can hop from one site to every arbitrary site of the lattice, we even permit fermionic interactions. Our main idea stems from an analogy with the classical statistical mechanics. The A.N.N.I. models has, for specific values of the coupling constants, an infinity of ground states. The dominant ground states are the ground states with the largest thermal fluctuations. If they are in finite number, they can give rise to the coexisting phases at low temperature. This was understood by Fisher and Selke,[6] and became a rigorous theory in the papers of Dinaburg, Mazel, and Sinai.[7, 8]

A similar idea was used in ref. 4 for the Falicov–Kimball model: the quantum fluctuations select, at low temperature, the "dominant quantum ground states," which are the classical ground states with the largest quantum fluctuations. In the case of small chemical potentials (in our notations), the Falicov–Kimball model has two dominant quantum ground

states, which give rise to the two coexisting phases at low temperature. We recover the results of refs. 2 and 3. This idea will be extended to the class **F**.

0.1. The Hamiltonians of the Class F on the Lattice Z^ν

$$\begin{aligned}
\mathbf{H}_V &= H_V^f + H_V^c \\
H_V^f &= \sum_{\{A,\, B \in \mathscr{P}_V \times \mathscr{P}_V\}} t_{\{A,\, B\}} \sum_{\{x \in V\}} C_{A+x}^* C_{B+x} \\
H_V^c &= \sum_{\{X \neq \varnothing,\, Y \neq \varnothing \in \mathscr{P}_V\}} J_{\{X,\, Y\}} \sum_{\{y \in V\}} \sigma_{X+y} \tau_{Y+y} \\
&\quad + \frac{\hat{U}}{2} \sum_{x \in V} \sigma(x)\, \tau(x) - \frac{\mu^i}{2} \sum_{x \in V} \sigma(x) - \frac{\mu^e}{2} \sum_{x \in V} \tau(x)
\end{aligned} \tag{0.1}$$

- $\mathscr{P}_V$ is the set of subsets of $V \subset Z^\nu$.
- C_a^*, C_a are the fermionic creation and annihilation operators acting on the usual Fock space.[10]
- $\tau_a = 2C_a^* C_a - 1$.
- σ_x is 1 if there is an ion at the site x and -1 otherwise.

$$C_A = \prod_{\{a \in A\}} C_a; \qquad \sigma_X = \prod_{\{x \in X\}} \sigma_x; \qquad \tau_Y = \prod_{\{y \in Y\}} \tau_y$$

- We suppose that $A \cap B = \varnothing$, and that $|A| = |B|$. ($|A|$ is the cardinality of A). The coupling constants $t_{\{A,\, B\}}$ and $J_{\{X,\, Y\}}$ are real numbers, which are translation invariant, they satisfy the following conditions:

$$\begin{aligned}
A! \times |t_{[A,\, B]}| &\leqslant \exp - D.T[A, B] \\
|J_{\{X,\, Y\}}| &\leqslant \exp - D'.T[X, Y]
\end{aligned} \tag{0.2}$$

D and D' are positive constants. We consider the set of trees, which can be built from the set $[A, B]$, we define $T[A, B]$ to be the length of the tree of minimal length. The factorial $A!$ is the cardinality of the different pairings between the creation and annihilation operators of the fermionic potential. $t_{a,\, b}$ is called the hopping intensity, $t_{[A,\, B]}$ is the hopping interaction. We will use the short term "hopping" for both. μ_e and μ_i are the chemical potentials of the electrons and resp. of the ions. We single out from the hamiltonian H^c the "on site energy" $\hat{U}$, which will play a major role. In the general case, $\hat{U}$ has an arbitrary sign, then we use the transformation $\sigma(x) \to -\sigma(x)$ at the sites for which $\hat{U} < 0$ to go to the repulsive case $\hat{U} > 0$

at every lattice site, that we will consider now. H^{int} is the hamiltonian containing the non on site potentials of H^C. The well known Falicov–Kimball hamiltonian belongs to the class **F**. In this case the electrons jump only on their nearest neighbors, more other there is no fermionic potential.

$$\mathbf{H}_V^{FK} = t \sum_{\langle a,b\rangle \in V} \{C_a^* C_b + C_b^* C_a\} + \frac{U}{2} \sum_{x \in V} \sigma(x)\, \tau(x) - \frac{\mu^i}{2} \sum_{x \in V} \sigma(x) - \frac{\mu^e}{2} \sum_{x \in V} \tau(x) \tag{0.3}$$

0.1.1. Definitions

- $\mathbf{\Sigma_V}$ is the set of ion's configurations Σ_V defined in V.
- $\mathscr{F}_V^f$ is the set of "frozen electron's" configurations in V (non moving).
- The local algebra $\mathscr{O}$ is the tensor product of the local algebra built from the fermionic operators and from the commutative local algebra $\mathscr{S}$ built from the $\sigma(x)$,[10] a typical element will be written $O \in \mathscr{O}$.
- The set of the boundary conditions (b.c.) $\mathscr{B}_{\bar{V}}$ with support in $\bar{V}$ (the overline means the complement of V in Z^ν) is the set of the tensor products $\Sigma_{\bar{V}} \otimes F_{\bar{V}}^f$, where $\Sigma_{\bar{V}} \in \mathbf{\Sigma}_{\bar{\mathbf{V}}}$, and $F_{\bar{V}} \in \mathscr{F}_{\bar{V}}^f$.
- The partition function, the free energy, the finite volume correlation functions, with the b. c. $\Sigma_{\bar{V}} \otimes F_{\bar{V}}^f$:

$$Z\{H; F_{\bar{V}} \otimes \Sigma_{\bar{V}}\} = \sum_{\Sigma_V} \operatorname{Tr}_{F_V} e^{-\beta H_V}$$

$$F(\{T\}, U, \mu^e, \mu^i, J_{A,B}) = -\lim_{V \to \infty} \frac{1}{\beta\, |V|} \ln\{Z\{H; F_{\bar{V}} \otimes \Sigma_{\bar{V}}\} \tag{0.4}$$

$$\langle O \rangle (\Sigma_{\bar{V}} \otimes F_{\bar{V}}) = \frac{\sum_{\Sigma_V} \operatorname{Tr}_{F_V} \{e^{-\beta H_V} \mathscr{O}\}}{Z\{H; F_{\bar{V}}^f \otimes \Sigma_{\bar{V}}\}} \tag{0.5}$$

The trace is taken over the Fock space in V, meanwhile $\Sigma_{\bar{V}} \otimes F_{\bar{V}}^f$ is fixed. The correlation functions are the weak limits of the finite volume correlation functions defined within the set of the b.c. $\mathscr{B}_{\bar{V}}$.

For sake of simplicity, we will use the following set of boundary conditions and the following sub-algebra for the correlation functions.

- An element of the set of the boundary conditions $\mathbf{\Sigma}_{\bar{\mathbf{V}}}$ in $\bar{V}$, is defined first by an ion's configuration $\Sigma_{\bar{V}} \in \mathbf{\Sigma}_{\bar{\mathbf{V}}}$, and secondly we sum over the (classical) configurations of frozen electrons $F_{\bar{V}}^f$. (Notice that the electrons coming from V can wander in $\bar{V}$.)

• The set of the finite volume correlation functions restricted to the sub-algebra $\mathscr{S}$ with b.c. $\Sigma_{\bar{V}}$:

$$\left\langle \prod_{x \in X} \sigma(x) \right\rangle^{\mathbf{H}} (\Sigma_{\bar{V}})$$

0.2. The Insulating Conditions

We want to find conditions on the hamiltonian $\mathbf{H}$, which imply, that the electrons are localized on some lattice sites. We notice that the effect of the quantum part H^f of the hamiltonian $\mathbf{H} \in \mathbf{F}$ is to move the electrons. The W.I. condition goes in the opposite direction: it introduces a barrier coming from the classical hamiltonian, to prevent the electrons from moving.

0.2.1. The Weak Insulating (W.I.) Condition

At first we impose two natural conditions leading to the insulating behavior:

(i) For each ion configuration Σ, there exists a configuration of frozen electrons F^f, which minimizes the hamiltonian H^c. The configuration $\{\Sigma, F^f\}$ is called a conditional ground state.

(ii) If one add or remove one or several different electrons from a conditional ground state $\{\Sigma, F^f\}$, there is a strictly positive energy gap K for the hamiltonian H^c per removed or per added electron.

The site energy at x is denoted $-H^c_x(\cdot, \cdot)$, where $(\cdot, \cdot)$ defines the configuration at x. The first argument is the ion's configuration, which is represented by • if there is an ion and by $\varnothing$ otherwise, the second argument describes the electronic configuration, which is represented by $-$ if there is an electron, and by $\varnothing$ otherwise. The dependence of $H^c_x(\cdot, \cdot)$ in the configuration outside of the site x is not written explicitly. We have to compute the four different expressions of $H^c_x(\cdot, \cdot)$

$$H^c_x(\varnothing, -) = -\frac{\mu_i}{2} + \frac{\mu_e}{2} - \frac{\hat{U}}{2} - \sum_{\{X, Y \mid x \in X; x \in Y\}} J_{\{X,Y\}} \sigma_{X/x} \tau_{Y/x}$$

$$+ \sum_{\{X, Y \mid x \notin X; x \in Y\}} J_{\{X,Y\}} \sigma_X \tau_{Y/x} - \sum_{\{X, Y \mid x \in X; x \notin Y\}} J_{\{X,Y\}} \sigma_{X/x} \tau_Y$$

$$H_x^c(\varnothing,\varnothing)=-\frac{\mu_i}{2}-\frac{\mu_e}{2}+\frac{\hat{U}}{2}+\sum_{\{X,Y\,|\,x\in X;\,x\in Y\}}J_{\{X,Y\}}\sigma_{X/x}\tau_{Y/x}$$
$$-\sum_{\{X,Y\,|\,x\notin X;\,x\in Y\}}J_{\{X,Y\}}\sigma_X\tau_{Y/x}-\sum_{\{X,Y\,|\,x\in X;\,x\notin Y\}}J_{\{X,Y\}}\sigma_{X/x}\tau_Y$$

$$H_x^c(\bullet,-)=\frac{\mu_i}{2}+\frac{\mu_e}{2}+\frac{\hat{U}}{2}+\sum_{\{X,Y\,|\,x\in X;\,x\in Y\}}J_{\{X,Y\}}\sigma_{X/x}\tau_{Y/x}$$
$$+\sum_{\{X,Y\,|\,x\notin X;\,x\in Y\}}J_{\{X,Y\}}\sigma_X\tau_{Y/x}+\sum_{\{X,Y\,|\,x\in X;\,x\notin Y\}}J_{\{X,Y\}}\sigma_{X/x}\tau_Y$$

$$H_x^c(\bullet,\varnothing)=\frac{\mu_i}{2}-\frac{\mu_e}{2}-\frac{\hat{U}}{2}-\sum_{\{X,Y\,|\,x\in X;\,x\in Y\}}J_{\{X,Y\}}\sigma_{X/x}\tau_{Y/x}$$
$$-\sum_{\{X,Y\,|\,x\notin X;\,x\in Y\}}J_{\{X,Y\}}\sigma_X\tau_{Y/x}+\sum_{\{X,Y\,|\,x\in X;\,x\notin Y\}}J_{\{X,Y\}}\sigma_{X/x}\tau_Y$$

(A) The condition (ii) provides two alternative conditions, if there is no ion at the site x:

$$H_x^c(\varnothing,\varnothing)-H_x^c(\varnothing,-)>k>0\Rightarrow-\mu_e+\hat{U}-2\sum_{\{X,Y\,|\,x\in Y\}}|J_{\{X,Y\}}|>K>0$$

$$H_x^c(\varnothing,-)-H_x^c(\varnothing,\varnothing)>K>0\Rightarrow\mu_e-\hat{U}-2\sum_{\{X,Y\,|\,x\in Y\}}|J_{\{X,Y\}}|>K>0$$

(B) The condition (ii) provides two alternative conditions, if there is an ion at the site x:

$$H_x^c(\bullet,-)-H_x^c(\bullet,\varnothing)>K\Rightarrow\mu_e+\hat{U}-2\sum_{\{X,Y\,|\,x\in X\}}|J_{\{X,Y\}}|>K$$

$$H_x^c(\bullet,\varnothing)-H_x^c(\bullet,-)>K\Rightarrow-\mu_e-\hat{U}-2\sum_{\{X,Y\,|\,x\in Y\}}|J_{\{X,Y\}}|>K$$

Note. These conditions do not involve the potentials depending of the ion's variables only.

The condition (ii) will be satisfied, if we combine one condition from (A) with one condition from (B).

(I) $H_x^c(\bullet,-)-H_x^c(\bullet,\varnothing)>K;\qquad H_x^c(\varnothing,\varnothing)-H_x^c(\varnothing,-)>K$

(II) $H_x^c(\bullet,\varnothing)-H_x^c(\bullet,-)>K;\qquad H_x^c(\varnothing,-)-H_x^c(\varnothing,\varnothing)>K$

(III) $H_x^c(\bullet,-)-H_x^c(\bullet,\varnothing)>K;\qquad H_x^c(\varnothing,-)-H_x^c(\varnothing,\varnothing)>K$

(IV) $H_x^c(\bullet,\varnothing)-H_x^c(\bullet,-)>K;\qquad H_x^c(\varnothing,\varnothing)-H_x^c(\varnothing,-)>K$

• Condition I (II): the set of the classical ground states is defined by the configurations containing at each site either an isolated electron or an isolated ion. (The conditions II is obtained from the condition I by using the transformation $\sigma_x \to -\sigma_x$.) So we will consider the case I.

• Condition III (IV). The conditions III and IV are related by symmetry. The set of the classical ground states is defined by the configurations containing at each site either an hole or an ion (either a pair or an isolated electron). There are no electrons to get quantum fluctuations. We have to add electrons to permit quantum fluctuations, then we will be outside the "half filled band." We will not considered this interesting case, which is analyzed in ref. 39.

Then the condition I (II) defines the weak insulating condition, which takes the form $U > 0$ ($U < 0$):

$$|\mu_e| < U - K; \qquad \text{with} \quad U := \hat{U} - 2 \sum_{\{X, Y \mid x \in Y\}} |J_{\{X, Y\}}| \tag{0.10}$$

Note. We believe that, in many cases the W.I. condition should be enough to prove, at least in part, the results contained in this paper. For the Falicov–Kimball model considered for $\mu_e = \mu_i = o$ (in our notations), the W.I. condition is reduced to $U > 0$. This is precisely the condition required by Kennedy and Lieb in ref. 2.

0.2.2. The Strong Insulating (S.I.) Condition

In this paper we need a stronger condition: the strong insulating (S.I.) condition, which mainly requires that the (classical) energy gap K prevails over the (quantum) hopping intensity of the electrons to prevent the transport of electrons. We need some definitions:

$$\|T\| = \sum_{\{0 \in A \subset Z^\nu\}} \left[\frac{A}{2}\right]! \times 2^{\frac{A}{2}} \times |t_A|$$

We suppose that $\|T\|$ is finite. Let $C > 1$ be a constant, next we define $\widetilde{U} =: C \times U$ and $\widetilde{\mu}^e =: C \times \mu^e$, notice that C and then $\widetilde{\mu}^e$ and $\widetilde{U}$ will change along this paper.

We will say that an S.I. condition is satisfied if:

$$\begin{aligned} &|r(\widetilde{\mu}^e)| < r(\widetilde{U}) - \|T\| - \exp - 2\beta\{r(\widetilde{U}) - |(\widetilde{\mu}^e)| - \|T\|\} + h.o. \\ &h.o. =: \|T\| \cdot O\left(\frac{\|T\|}{r(\widetilde{U})}\right) + O(\exp - 2\beta\{r(\widetilde{U}) - |r(\widetilde{\mu}^e)| - \|T\|\}) \end{aligned} \tag{0.11}$$

Definition. $\mathscr{D}(\mathscr{H})$ is the manifold defined by the coupling constants of $\mathbf{H} \in \mathbf{F}$ satisfying an S.I. condition.

0.3. The Effective Hamiltonian

Next we define the class of the generalized Ising models, which contains the effective hamiltonians of $\mathbf{F}$.

0.3.1. The Generalized Ising Models I

$$H_V^I = \sum_{\{x \in V,\, X \subset \mathscr{P}_V\}} J_X \prod_{\{y \in (X+x)\}} \sigma_y \tag{0.6}$$

We suppose that there exists a positive constant D such that:

$$|J_X| \leqslant \exp - D'.T[X] \tag{0.7}$$

it is easy to show that there exists a constant $D'_m > 0$, such that if $D' > D'_m$:

$$\sum_{\{X \in Z^\nu \mid o \in X\}} |J_X| \leqslant \infty \tag{0.8}$$

then, under the condition (0.8), the Gibbs states are the solutions of the D.L.R. equations.

$$E^{H^I}[\sigma_X \mid \Sigma_{\bar{V}}] = \frac{\sum_{\Sigma_V} e^{-\beta H^I[\Sigma_V \mid \Sigma_{\bar{V}}]} \sigma_X(\Sigma_V)}{\sum_{\Sigma_V} e^{-\beta H^I[\Sigma_V \mid \Sigma_{\bar{V}}]}} \tag{0.9}$$

0.3.2. The Effective Hamiltonians

Our goal is to give a rigorous meaning to the following formula, which follows from the idea of ref. 2.

$$Z\{H; \Sigma_{\bar{V}}\} = \sum_{\Sigma_V} [\mathrm{Tr}_{F_V \otimes F_{\bar{V}}^f} e^{-\beta H}] =: \sum_{\Sigma_V} e^{-\beta \mathscr{H}_\beta(\Sigma_V \mid \Sigma_{\bar{V}})}$$

where the effective hamiltonian $\mathscr{H}_\beta \in \mathbf{I}$ is built from a cluster expansion expressed in term of the weights of the loops, which represent the quantum fluctuations compatible with an ion configuration Σ_V.

0.3.3. Definition.

The hamiltonian $\mathbf{H} \in \mathbf{F}$ admits an effective hamiltonian $\mathscr{H}_\beta$ if:

(I) $\mathscr{H}_\beta \in \mathbf{I}$

(II) for any finite volume V, for any set $X \in V$, and for any boundary condition $\Sigma_{\bar{V}} \in \mathbf{\Sigma}_{\bar{\mathbf{V}}}$ the following identities hold:

$$\langle \sigma_X \rangle^{\mathrm{H}} (\Sigma_{\bar{V}}) = E^{\mathscr{H}_\beta}[\sigma_X \mid \Sigma_{\bar{V}}] \tag{0.12}$$

0.3.4. Definition

The "p order decomposition" of $\mathscr{H}_\beta(V)$ is:

$$\begin{aligned} \mathscr{H}_\beta(V) &= \mathscr{H}^{o}_{\beta}(V) + \cdots + \mathscr{H}^{p}_{\beta}(V) + \sum_{i=p+1}^{i=\infty} \mathscr{H}^{j}_{\beta}(V) \\ &=: \mathscr{H}^{\leqslant p}_{\beta}(V) + \mathscr{H}^{> p}_{\beta}(V) \end{aligned} \tag{0.13}$$

The truncated hamiltonian $\mathscr{H}^{p}_{\beta}(V)$ is defined by the potentials for which p hoppings factorize. The main point is that the potentials contained in the tail hamiltonian $\mathscr{H}^{> p}_{\beta}(V)$ decay exponentially. Notice that the truncated effective potentials depend explicitly of the temperature. The main advantage of this decomposition is that it is valid at every temperature, one could study the existence of intermediate phases as well as the critical properties of the model. The main inconvenient is that the computations of the truncated effective hamiltonians are rapidly complicate. We define a more convenient decomposition of $\mathscr{H}_\beta(V)$ relevant at low temperature only.

0.3.5. Definition

The low temperature (L.T.) p order decomposition of $\mathscr{H}_\beta(V)$
Each q order effective hamiltonian ($q \leqslant p$) splits into two parts:

$$\mathscr{H}^{q}_{\beta}(V) := \mathscr{H}^{q}_{\infty}(V) + \mathscr{H}^{q}_{\{w,\beta\}}(V)$$

where the potentials contained in $\mathscr{H}^{q}_{\infty}(V)$ are independent of the temperature, they are obtained by letting $\beta \to \infty$ in the potentials of $\mathscr{H}^{q}_{\beta}(V)$. $\mathscr{H}^{q}_{\{w,\beta\}}(V)$ are the temperature dependent potentials. The L.T. p order decomposition of the effective hamiltonian is given by:

$$\begin{aligned} \mathscr{H}_\beta(V) &= \mathscr{H}^{o}_{\infty}(V) + \cdots + \mathscr{H}^{p}_{\infty}(V) + \mathscr{H}^{\leqslant p}_{\{w,\beta\}}(V) + \mathscr{H}^{> p}_{\beta}(V) \\ \mathscr{H}^{\leqslant p}_{\{w,\beta\}}(V) &:= \mathscr{H}^{o}_{\{w,\beta\}}(V) + \cdots + \mathscr{H}^{p}_{\{w,\beta\}}(V) \end{aligned} \tag{0.14}$$

The effective hamiltonians $\mathscr{H}^{p}_{\infty}$ are easy to compute from the loop's representation developed in this paper, or in some cases by some other methods.[35, 36] $\mathscr{H}^{\leqslant p}_{\{w,\beta\}}(V)$ and $\mathscr{H}^{> p}_{\beta}$ act as a perturbation of $\mathscr{H}^{\leqslant p}_{\infty}$.

Theorem. We consider the class of the hamiltonians $\mathbf{H} \in \mathbf{F}$, which satisfy an S.I. condition (0.11).

(I) The hamiltonian $\mathbf{H}$ admits, for every temperature, an effective Hamiltonian $\mathscr{H}_\beta$.

(II) There exists a constant $A_p > 0$ such that, if $\beta \frac{\|T\|^{p+1}}{[\tilde{U} - \tilde{\mu}^e]^p} > A_p$, $\mathscr{H}_\beta$ admits a low temperature p order decomposition.

Note. The above approach can be generalized as follows: the set of random variables σ_v can be replaced by more general r.v. such as a family of gaussian r.v., in this case we get the static Holstein model.[27]

0.4. The Hierarchy of the Phase Diagrams: Pirogov–Sinai Theory

Next we construct the phase diagram of the models via the classical Pirogov–Sinai (P.S.) theory,[14, 15] extended to the long range potential hamiltonians in refs. 16a, 16b, and 17. The phase diagram is obtained through a hierarchical construction starting from the zeroth order decomposition of the effective hamiltonian, and by using the zeroth order P.S. theory. The phase diagram will be refined by going to higher order.

Definitions. The "l shrinked domain" is defined by: $S_{l^p}(B) =: \{x \in B \subset R^n \mid d(x, \bar{B}) < l^p\}$. In the following we will have to consider $l = \frac{1}{U - |\mu^e| - \|T\|}$, hereafter we will use the short notation $S_p(B)$ for $S_{l^p}(B)$.

0.4.1. The Zero Order P.S. Theory

• We start from the zero order L.T. decomposition of the effective hamiltonian $\mathscr{H}_\beta$.

$$\mathscr{H}_\beta(V) =: \mathscr{H}^o_\infty(V) + \mathscr{H}^o_{\{w,\beta\}}(V) + \mathscr{H}^{>o}_\beta(V) \tag{0.15}$$

$$\mathscr{H}^o_\infty(V) = \sum_{\{X, Y \in \mathscr{P}_V \times \mathscr{P}_V\}} J_{\{X,Y\}} \sum_{\{y \in V\}} (-)^{|Y|} \sigma_{\{X \cdot (Y+y)\}} - \frac{\mu^i - \mu^e}{2} \sum_{x \in V} \sigma(x) \tag{0.16}$$

where $X \cdot Y = X \cup Y - X \cap Y$. We define the family of sub-manifolds $\{\mathscr{D}^{[o,a_o]}\}_{a_o \in M_o} \subset \mathscr{D}(\mathscr{H})$ in which the P.S. theory can be applied to $\mathscr{H}^o_\infty$, the family is not empty, because for $|\mu^e - \mu^i|$ large, the P.S. theory applies. These manifolds are generally separated by hyper-surfaces, along which $\mathscr{H}^o_\infty$ is infinitely degenerated. Then we will prove that, the low temperature zero order phase diagram is a smooth deformation of the phase diagram of $\mathscr{H}^o_\infty$, if the coupling constants of $\mathbf{H}$ belong to the shrinked domains $\{S_1(\mathscr{D}^{[o,a_o]})\}_{a_o \in M_o}$.

The model behaves like its classical part: this is the case I.

The phase diagram of $\mathbf{H} \in \mathbf{F}$ remains unknown in $\mathscr{D}(\mathscr{H}) / \{\bigcup_{a_o} S_o(\mathscr{D}^{[o, a_o]})$. We have to go to the next order. More generally we will suppose that the phase diagram of $\mathbf{H} \in \mathbf{F}$ has been constructed up to the order $p-1$, this means, that we have built the phase diagram of $\mathbf{H}$ in the family of the shrinked domains $\{S_{p-1}(\mathscr{D}^{[p-1, a_{p-1}]})\}_{a_{p-1} \in A_{p-1}}$. Next we construct the p order P.S. theory.

0.4.2. The p Order P.S. Theory

• We start from the p order L.T. decomposition of $\mathscr{H}_\beta$.

$$\mathscr{H}_\beta(V) = \mathscr{H}_\infty^{\leqslant p}(V) + \mathscr{H}_{\{w, \beta\}}^{\leqslant p}(V) + \mathscr{H}_\beta^{> p}(V) \tag{0.17}$$

We define the family of sub-manifolds $\{\mathscr{D}^{[p, a_p]}\}_{a_p \in A_p} \subset \mathscr{D}(\mathscr{H})$ in which the P.S. theory can be applied to $\mathscr{H}_\infty^{\leqslant p}$, this family contains the shrinked sub-manifolds $\{S_{p-1}(\mathscr{D}^{[p-1, a_{p-1}]})\}_{a_{p-1} \in M_{p-1}}$ constructed from the $p-1$ order. These manifolds are generally separated by hyper-surfaces, along which $\mathscr{H}_\infty^{\leqslant p}$ is infinitely degenerated. Then we will prove that the p order phase diagram of $\mathbf{H}$ is a smooth deformation of the phase diagram of $\mathscr{H}_\infty^{\leqslant p}$ in the shrinked domains $S_p(\mathscr{D}^{[p, a_p]})$. Then, in general, two situations occur at the order p.

The domains of phase coexistence, defined at the order $p-1$, are enlarged,

The most interesting case: new quantum phase transitions appear in new shrinked domains, this is the case II.

Corollary. The hierarchical structure of the phase diagram of $\mathbf{H} \in \mathbf{F}$.

The hamiltonian $\mathbf{H} \in \mathbf{F}$ satisfies an S.I. condition (0.11). $A^{[o, a(o)]}$, $A^{[p, a, (p)]}$ are positive constants.

(1) The zero order P.S. theory: for $\beta > a^{[o, a(o)]}$, the phase diagram of $\mathbf{H}$, which is obtained at the zero order, is a smooth deformation of the phase diagram of the truncated hamiltonian $\mathscr{H}_\infty^o$, if the coupling constants of $\mathbf{H}$ belong to the shrinked domains $\{S_1(\mathscr{D}^{[o, a_o]})\}_{a_o \in M_o} \subset \mathscr{D}$.

(2) The p order P.S. theory: for $\beta \frac{\|T\|^{p+1}}{[\tilde{U}_i - \tilde{\mu}^e]^p} > A^{[p, a(p)]}$. Then, for every $a_p \in M_p$, the phase diagram of $\mathbf{H}$, which is obtained at the p order, is a smooth deformation of the phase diagram of the effective hamiltonian $\mathscr{H}_\infty^{\leqslant p}$ in the family of shrinked domains $\{S_{p+1}(\mathscr{D}^{[p, a_p]})\}_{a_p \in M_p} \subset \mathscr{D}$.

The paper is organized as follows. Section 1 deals with the loop's representation. Section 2 contains the construction of the effective hamiltonian. Section 3 contains the Pirogov–Sinai theory. The last section contains the technical results:

- Appendix A: the estimates on the contributions of the loops for $J_{\{X,Y\}} = 0$.
- Appendix B: the Dobrushin's inequalities.
- Appendix C: the cluster expansion for the conditional partition functions.
- Appendix D: the construction of the effective hamiltonian.
- Appendix E: the extension to the case $J_{X,Y} \neq 0$.

1. THE LOOPS FEYNMAN–KAC REPRESENTATION

We will construct a twofold Feynman–Kac (F.K.) representation for the matrix elements of the operator $\exp(-\beta \mathbf{H}_V)$, where $\mathbf{H} \in \mathbf{F}$.

1.1. The Feynman–Kac Representation via the Electron's Trajectories

We describe the usual F.K. representation,[10] which is expressed in term of the electron's trajectories, extended to the case of fermionic potentials (Fig. 1). We define the extended space time $\Lambda = V \times [0, \beta)$, in which the hyperplanes $t = o$ and $t = \beta$ are identified, this hyperplane V is called the basis. We fix the orthonormal basis in $\bar{\mathscr{K}}_V$ labelled by the pairs (F_V^f, Σ_V) of classical $\{0, 1\}$-configurations in V, and by the boundary conditions $\Sigma_{\bar{V}} \otimes F_{\bar{V}}$ in $\bar{V}$. By $\mathscr{S}(F'_V, F''_V)$, we denote the set of all one-to-one mappings between the electronic configurations F'_V, F''_V with $|F'_V| = |F''_V|$ and by par π the parity of the mapping $\pi: F'_V \to F''_V$. $D((F'_V \otimes \Sigma_V), (F''_V \otimes \Sigma_V) \mid \Sigma_{\bar{V}} \otimes F_{\bar{V}}^f)$ is the matrix element of $\exp(-\beta \mathbf{H}_V)$ in the reference basis:

$$
\begin{aligned}
&D((F'_V \otimes \Sigma_V), (F''_V \otimes \Sigma_V) \mid \Sigma_{\bar{V}} \otimes F_{\bar{V}}^f) = 0, \qquad \text{if} \quad |F'_V| \neq |F''_V| \\
&D((F'_V \otimes \Sigma_V), (F''_V \otimes \Sigma_V) \mid \Sigma_{\bar{V}} \otimes F_{\bar{V}}^f) \\
&\quad = \sum_{\pi \in \mathscr{S}(F'_V, F''_V)} (-1)^{\text{par } \pi} \times \prod_{j \in F'_V} \left(\int_{M_{j,\pi(j)}} P_{j,\pi(j)}(d\omega_j) \right) \\
&\qquad \times \exp - \beta \mathscr{H}^o_\infty(\Sigma_V \mid \Sigma_{\bar{V}}) \times \exp \left[-(U + \mu^e) \sum_{j \in V} T(\bullet_j, -_j) \right] \\
&\qquad \times \exp \left[-(U - \mu^e) \sum_{k \in V} T(\varnothing_k, \varnothing_k) \right] \\
&\qquad \times \exp \left\{ - \int_o^\beta \{ \mathscr{W}^s(\Sigma_V \mid \Sigma_{\bar{V}}) - \mathscr{W}^o(\Sigma_V \mid \Sigma_{\bar{V}}) \} \, ds \right\} \\
&\qquad \times V[\omega_1 \cdots \omega_{|F'_V|}] \qquad \text{if} \quad |F'_V| = |F''_V|
\end{aligned}
\tag{1.1}
$$

The first term stems from the integration over the electron's trajectories. $M_{j,j'}(=M^{\beta}_{j,j'})$ is the set of right-continuous paths ω of electron's trajectories of length time β in $\mathbf{Z}^{\nu}$ with $\omega(0)=\lim_{s\to 0}\omega(s)=j$, $\omega(\beta)=\lim_{s\to\beta}\omega(s)=j'$, and $P_{j,j'}(=P^{\beta,\Lambda}_{j,j'})$ denotes the path distribution restricted to $M_{j,j'}$ of the Markov jump process on $\mathbf{Z}^{\nu}$ with jump's intensity $x\to y$ given by $t_{x,y}$ (for the theory of these processes see, e.g., ref. 32). The conditional classical ground states are defined by an ion's configuration $[\Sigma_V \mid \Sigma_{\bar V}]$ together with the set of equations $\sigma_x\cdot\tau_x=-1$, for all $x\in V$. Next we extract the zero order effective hamiltonian $\mathscr{H}^{o}_{\infty}$ explicitly written in (0.16). $T(\bullet_{j,-j})$ is the time which the path ω_j and the vertical ion line spend together above the site j. $T(\varnothing_k,\varnothing_k)$ is the time during which, a hole is living above the site k. The next fourth terms are a convenient decomposition of the classical energy density computed at the time s. First we extract from $\mathscr{H}^{o}_{\infty}[\Sigma_V \mid \Sigma_{\bar V}]$ the term $\mathscr{W}^{o}(\Sigma_V \mid \Sigma_{\bar V})$, which is the non on site part of $H^{o}(\Sigma_V \mid \Sigma_{\bar V})$. $\mathscr{W}^{s}(\Sigma_V \mid \Sigma_{\bar V})$ is the classical energy density, computed at the time s, for a given configuration in the space time. According to the b.c., the electron's trajectories can wander in $\bar\Lambda$, provided that they intersect Λ.

The fermionic potential $V[\omega^{e}_1,\ldots,\omega^{e}_p]$ is a function of the electron's trajectories $[\omega^{e}_1,\ldots,\omega^{e}_n]$. The action of a monomial $t_{\{(a_o,b_o);\ldots;(a_n,b_n\}}C^{*}(a_o)\times C(b_0),\ldots,C^{*}(a_n)\,C(b_n)$ is to attribute the "hopping" $t_{\{(a_o,b_o);\ldots;(a_n,b_n\}}$ to the simultaneous jumps occurring on the set of bonds $\{(a_o,b_o),\ldots,(a_n,b_n)\}$. The electron's trajectories containing these simultaneous jumps are connected by links. A maximal set of linked trajectories build a linked family of electron's trajectories. Notice that every configuration with a family of jumps occurring at the same time has zero probability, as far as they do not come from a fermionic interaction. The action of the fermionic interaction hamiltonian on the space of the electron's trajectories gives a "positive probability" to simultaneous jumps. The resealed hopping intensity of a set of linked jumps occurring on the set of bonds $\{(a_o,b_o),\ldots,(a_n,b_n)\}$ is the resealed intensity $\frac{t\{a_o,b_o)\cdots(a_n,b_n)\}}{t(a_o,b_o)\times\cdots\times t(a_n,b_n)}$. In addition, to identify the individual jump's times of the electrons on the bonds of $\{(a_o,b_o),\ldots,(a_n,b_n)\}$, we insert the products of the δ functions defined on the pairwise individual hoppings' times of the linked trajectories, which identify the jump's times of the simultaneous jumps. The fermionic potential $V[\{\omega^{e}_1,\ldots,\omega^{e}_p\}]$ is the product of the resealed hoppings of the linked trajectories, and of the products of the δ "functions."

1.2. The Feynman–Kac Representation in Term of Loops

Next we construct the loop's representation, which was first introduced in ref. 11 for the case of the Hubbard model. A similar representation was introduced in ref. 12 for the one dimension quantum Heisenberg model.

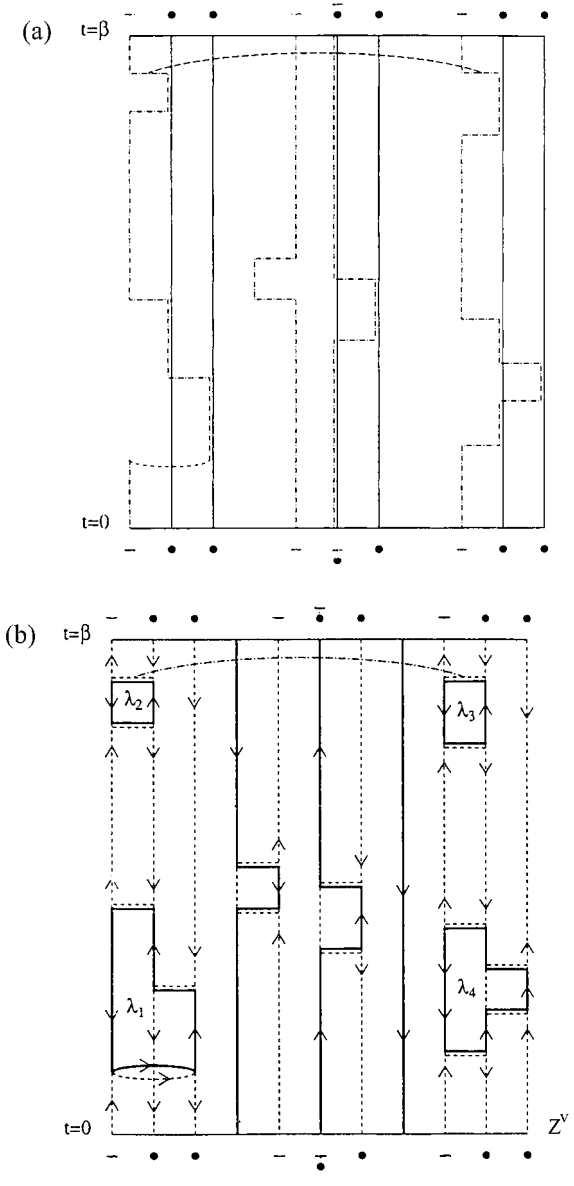

Fig. 1. (a) A configuration in term of the ions lines and in the lectrons trajectories; (b) the corresponding loops' configuration. The loop λ_1 contains an electron's jump of length 2, the loops λ_2 and λ_3 build a linked loop.

1.2.1. The Geometric Construction of the Loops (Fig. 1bis)

One of the following four situations occurs.

Case (i) There is an electron: we draw on the vertical lines a dashed line with an up arrow.

Case (ii) There is an ion: we draw a dashed line with a down arrow.

Case (iii) There is a pair: we draw a continuous segment with an up arrow.

Case (iv) There is a hole: we draw a continuous segment with a down arrow.

We complete this construction with additional curves. When an electron hops from the site x to the site y, the points x to y are connected by an horizontal curve of the same type and with the same arrow as the vertical lines coming at the sites x and y. A configuration is now described by a family of continuous loops and a family of dashed loops. All the vertical lines are covered with parts of oriented loops, two loops of different kinds meet along the additional lines of the loops, and secondly that some loops may wind around the torus. Notice that two loops of the same kind cannot intersect along the vertical parts except at the boundary points of their vertical segments, they are said to be compatible. The loops are characterized by their jump's times. The time origin s_o is the F.K. time of occurrence of an arbitrary jump of the loop, the other jumps' times $\{s_1,\ldots, s_n\}$, are counted along the orientation of the loop.

Next we classify the continuous loops according to their homotopical properties.

- The continuous non winding loops are denoted by $\mathscr{L}^o$.
- The winding continuous loops $\mathscr{L}^w$ are the up arrows winding continuous loops (resp. the loops $\mathscr{L}^{-w}$ down arrows loops) with positive winding number w (resp. with negative winding number $-w$). The loops $\mathscr{L}^w_r$ are the loops of $\mathscr{L}^w$ with r jumps. The winding loops of $\mathscr{L}^1_o$ have winding number $+1$ and -1, they are defined by their orientations and by the point of the loop with the basis.

Note. The loops interact in two ways: first through the fermionic potentials, then we construct the linked loops, secondly through the classical potentials, then we construct the interacting linked loops.

1.2.2. The Linked Continuous Loops (Fig. 1bis)

The simultaneous jumps occur either in the same loop, or in different loops by the fermionic potentials.

• The support of a simultaneous jump is composed of a pair of sets of vertices $\{B, C\}$, where $B = \{b_1^i,..., b_p^i\}$ is the support of the creation operators of the fermionic potential, and $C = \{c_1^i,..., c_p^i\}$ is the support of the annihilation operators. We have $|B| = |C|$.

• A pairing is a one to one correspondence between the elements of B and the elements of C. In this case there exists $p!$ different pairings, for simplicity we will identify the pair $\{B, C\}$ to a single letter A, and a pairing of the pair with a pairing of A^P. A family of pairings is written $\mathscr{A}^{\mathscr{P}} = \{A_1^{P_1},..., A_n^{P_n}\}$. We draw a set of links between the simultaneous jumps of the loops, which correspond to a pairing (there is not a unique choice),

• A linked continuous loop $\hat{\lambda}$ is a set of loops maximally connected by links.

• A non winding linked continuous loop $\hat{\lambda}$ is defined by two sets:

(i) a family of even sets of vertices with a pairing $\mathscr{A}^{\mathscr{P}} = \{A_1^{P_1},..., A_n^{P_n}\}$,

(ii) a sequence of jump's times composed of an arbitrary birth time s_o counted in the F.K. time, the other jumps' times $\{s_1,..., s_{n-1}\}$, one per simultaneous jump, are ordered along the orientation of the each constituent loop of $\hat{\lambda}$, (a detailed construction of the jumps' times is given in the Appendix A).

• A winding linked continuous loop is defined by the two sets defined above, supplemented by the family of the winding numbers of the constituent winding loops of $\hat{\lambda}$.

We will need a refined decomposition of the winding loops λ^W. Any time section of a winding loop $\lambda^w \in \hat{\lambda}^W$ by the hyperplane $s = \tau$, is crossed by $w+p$ up arrow continuous lines and by p down arrow continuous lines, p depends of τ, while w does not. Let $(x_o, 0)$ be one intersection of λ^w with the basis. The part of the loop in between $(x_o, 0)$ and the first relative maximum w.r.t. the time, defined by the space time point (x_1, s_1) is called a reduced part, the corresponding jump is called a turning jump. After the time s_1, the loop wanders until it reaches again the hyperplane defined by $s = s_1$, the corresponding part of λ^w is called a bubble part. We iterate this procedure until each constituent winding loop of λ^W has been totally decomposed into a family of disjoint standard parts and a family of disjoint bubble parts (Figs. 2a and b).

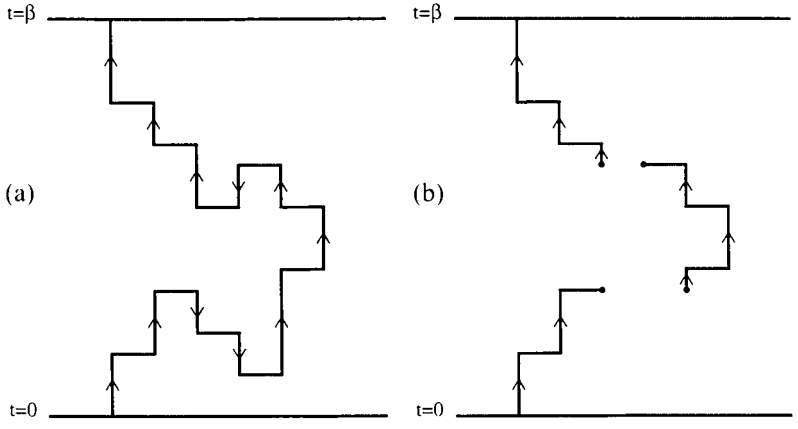

Fig. 2. (a) A windind loop; (b) the same winding loop in which the reduced part appears alone.

1.2.3. The Interacting Linked Continuous Loops

The classical hamiltonian creates, at each time s, either interactions between different loops, which are at distance less that l, or self interactions in a loop for pieces of the loop at distance smaller than l. The corresponding rescaled energy density is defined by:

$$\tilde{\mathscr{W}}^s(\Sigma_V \mid \Sigma_{\bar{V}}) := \mathscr{W}^s(\Sigma_V \mid \Sigma_{\bar{V}}) - \mathscr{W}^o(\Sigma_V \mid \Sigma_{\bar{V}})$$

The time's section of the space time by the hyperplane V_s of a loop's configuration is composed of vertices which are the intersections of V_s with the continuous up (resp. down) arrows, the corresponding vertices are called up (resp. down) interacting vertices. The energy density $\tilde{\mathscr{W}}^s(\Sigma_V \mid \Sigma_{\bar{V}})$ is non zero if the hyperplane V_s contains up or down interacting vertices at distance less than 1.

Definitions.

- Two segments $\{x, [s_1, s_2]\}$ and $\{y, [s_1, s_2]\}$ belonging to two continuous segments, which are either in the same continuous loop (self interaction), or in two different continuous loops interact through H^c, if $x - y \leqslant l$ and if $[s_1, s_2]$ is of maximal length.
- The distance $d(\hat{\lambda}_1; \hat{\lambda}_2)$ is the minimal distance between the vertical segments of the loops $\hat{\lambda}_1$ and of the loops $\hat{\lambda}_2$ computed at the same time.
- Two loops interact if they interact through at least two vertical segments.

• An interacting linked loop λ^I is a maximal connected set of linked loops $\{\hat{\lambda}_1,\ldots,\hat{\lambda}_p\}$ which interact.

1.2.4. The Loop's Space $\mathscr{L}\{H;\Lambda\}$

We will use the generic term of "loop" for the continuous interacting linked loop unless it needs to be specified. We have established a one to one correspondence between two different representations of a quantum configuration.

(1) An ion's configuration together with a family of interacting electron's trajectories.

(2) Two families of compatible oriented interacting linked loops composed of a family of dashed loops and a family of continuous loops.

• The classical ground states are represented by vertical dashed winding loops, each winding loop has a winding number which is equal to ± 1.

• The continuous interacting linked loops represent the quantum fluctuations, they will be the polymers used to build the cluster expansion.

Definition. $\mathscr{L}\{H;\Lambda\}$ is the set of loops intersecting Λ, which can be built from **H**.

Note. The study of off diagonal correlation function needs a simple extension of the loop's representation: we introduce open loops which are sticked to the basis. For example $\langle C_x^* C_y \rangle$ is represented by mean of open loops starting from x and ending at y. We refer to ref. 34 for more details in the case of the FK model.

1.2.5. The Signed Densities (s.d.) of the Loops

First notice that the term "signed density" is a convenient abuse of language. We recall and extend the essential properties of the loop's space proved in ref. 4.

• Each non linked non interacting loop λ is endowed with an intrinsic sign $\epsilon(\lambda)$ due to the Fermi statistics of the electrons. For completeness we sketch the proof. Every electronic configuration defines the sign $(-1)^{\text{par }\pi}$ (1.1), which factorizes into the product of the signs attached to each loop of the configuration, the main reason is that, the electrons located "above" and "below" a given loop, belong to the same set, but are generally permuted. The parity of this permutation defines the sign attached to the loop.

• The sign of the s.d. of a linked interacting loop $\hat{\lambda}^I$ is the product of the signs of the s.d. of the constituent loops of $\hat{\lambda}^I$.

- The s.d. of the non winding loops do not depend of the chemical potential, because the vertical lengths of the up arrow lines and of the down arrow lines are equal, meanwhile the s.d. of the winding loops do. We write successively the s.d. of:

- a non winding non interacting loop λ,
- a winding non interacting loop $\lambda^{\pm w}$ with either a positive or a winding number w,
- a non winding linked loop $\hat{\lambda}$ built from the family of non winding linked loops $\{\lambda_1,\ldots, \lambda_q\}$,
- a winding linked non interacting loop $\hat{\lambda}^w$,
- a general linked interacting loop $\hat{\lambda}^I$, which is built from the family of linked non winding loops $\{\hat{\lambda}_1,\ldots, \hat{\lambda}_p\}$ and from the family of winding linked loops $\{\hat{\lambda}_1^w,\ldots, \hat{\lambda}_q^w\}$.

$$\varphi(\lambda) = \epsilon(\lambda)\, t_{(x_o, x_1)} \times \cdots \times t_{(x_n, x_o)} \chi(s_1,\ldots, s_n) \times \exp -\hat{U} \sum_{i=o}^{i=n-1} m_i(s_{i+1} - s_i) + \tilde{\mathscr{W}}^{s_1,\ldots, s_n}(\lambda)$$

$$\varphi(\lambda^{\pm w}) = \epsilon(\lambda^{\pm w}) \times t_{(x_o, x_1)} \times \cdots \times t_{(x_n, x_o)} \times \chi(s_1,\ldots, s_n) \times \exp\left\{ -\beta\, |w| \{\hat{U} \pm \mu^e\} - \hat{U} \sum_{i=o}^{i=n-1} m_i(s_{i+1} - s_i) \right\} \times \exp\{\tilde{\mathscr{W}}^{s_1,\ldots, s_n}(\lambda^{\pm w})\}$$

$$\varphi(\hat{\lambda}) = t_{A_o} \times \cdots \times t_{A_n} \times \prod_{j=1}^{j=p} \epsilon(\lambda_j) \times \exp\left\{ -\hat{U} \sum_{i=o}^{i=n-1} m_i(s_{i+1} - s_i) \right\} \times \chi(s_1,\ldots, s_n) \times \exp\{\tilde{\mathscr{W}}^{s_1,\ldots, s_n}(\lambda_1,\ldots, \lambda_q)\} \tag{1.2}$$

$$\varphi(\hat{\lambda}^w) = t_{A_o} \times \cdots \times t_{A_n} \times \left[\prod_{j=1}^{j=r} \epsilon(\lambda_j) \exp\left\{ -\hat{U} \sum_{i=o}^{i=n-1} m_i(s_{i+1} - s_i) \right\} \times \chi(s_o,\ldots, s_n) \right] \times \left[\prod_{k=1}^{k=s} \epsilon(\lambda_k^{w_k}) \exp\{-\beta w_k(\hat{U} + \mu^e)\} \right] \times \left[\prod_{l=1}^{l=u} \epsilon(\lambda_l^{-w'_l}) \exp\{-\beta w'_l(\hat{U} - \mu^e)\} \right] \times \exp\{\tilde{\mathscr{W}}^{s'_1,\ldots, s'_n}(\lambda^{\pm w})\}$$

$$\varphi(\hat{\lambda}^I) = \left[\prod_{i=1}^{i=p} \varphi(\hat{\lambda}_i) \times \prod_{j=1}^{j=q} \varphi(\hat{\lambda}_j^w) \right] \times \exp \tilde{\mathscr{W}}(\hat{\lambda}_1,\ldots, \hat{\lambda}_p; \hat{\lambda}_1^w,\ldots, \hat{\lambda}_q^w)$$

- $\{s_o; s_1,\ldots, s_n\}$ are the jump's times of the loop, s_o is the F.K. time of a jump, which is taken as the origin, $\chi(s_1 < \cdots < s_n)$ is the characteristic function of the ordered jump's times counted along the orientation of the loop with origin s_o.

- The positive integers m_i are the numbers of vertical segments contained in the non winding linked loops between the times s_{i+1} and s_i, and the numbers of vertical segments contained in the bubble parts of the winding linked loops between the consecutive times s_{i-1} and s_i.

- $\tilde{\mathscr{W}}^{s'_1,\ldots,s'_n}(\lambda)$ is the sum of the energy densities piecewise constant between two successive jumps' times of the loop λ (here the jumps' times s'_i are the Feynman–Kac times).

- $\tilde{\mathscr{W}}(\hat{\lambda}_1,\ldots,\hat{\lambda}_p;\hat{\lambda}^w_1,\ldots,\hat{\lambda}^w_q)$ is the integrated interactions' energy densities and self energy densities of the linked interacting loop built from the loops $\{\hat{\lambda}_1,\ldots,\hat{\lambda}_p;\hat{\lambda}^w_1,\ldots,\hat{\lambda}^w_q\}$.

1.3. Loop's Representation of the Correlation Functions

We define the basic notion of conditional ensemble. The conditional ensembles, which are the quantum analogs of the restricted ensembles introduced in the classical statistical mechanics.[9]

1.3.1. Definitions

- The conditional ensemble $\mathscr{COND}(H; \Sigma_V \mid \Sigma_{\bar{V}})$ is the subset of the loops of $\mathscr{L}\{H; \Lambda\}$, which are compatible with the ion's configurations $\Sigma_{\bar{V}}$ in V, and $\Sigma_{\bar{V}}$ in $\bar{V}$

- The conditional partition function $Z\{H; \Sigma_V \mid \Sigma_{\bar{V}}\}$ is the restriction of the partition function, associated to the hamiltonian $\mathbf{H}$, to the conditional ensemble $\mathscr{COND}(H; \Sigma_V \mid \Sigma_{\bar{V}})$.

1.3.2. Proposition

The conditional partition functions, the partition function, and the correlation functions, defined from the hamiltonian $\mathbf{H}$, are expressed in the loop's F.K. representation:

(i) $$Z\{H; \Sigma_V \mid \Sigma_{\bar{V}}\} = \exp -\beta \mathscr{H}^o_\infty(\Sigma_V \mid \Sigma_{\bar{V}}) \times \left[\sum_{\{\hat{\lambda}^I_1 \cdots \hat{\lambda}^I_r\} \subset \mathscr{COND}(H; \Sigma_V \mid \Sigma_{\bar{V}})} \prod_{i=1}^{r} \varphi(\hat{\lambda}^I_i) \right]$$

$$\text{(ii)} \quad Z(H; \Sigma_{\bar{V}}) = \sum_{\Sigma_V} Z(H; \Sigma_V \mid \Sigma_{\bar{V}})$$

$$\text{(iii)} \quad \langle \sigma_X \rangle^H (\Sigma_{\bar{V}}) = \frac{\sum_{\Sigma_V} Z\{H; \Sigma_V \mid \Sigma_{\bar{V}}\}\, \sigma_X(\Sigma_V)}{Z(H; \Sigma_{\bar{V}})} \tag{1.3}$$

Proof. The representation (i) is the transcription of the F.K. representation, previously written in term of electron's trajectories, into the F.K. representation written in term of linked interacting loops (polymers). The representations (ii) and (iii) are trivial consequences of (i). ▌

2. THE EFFECTIVE HAMILTONIANS

We will show that every hamiltonian **H** which satisfies an S.I. condition, admits an effective hamiltonian $\mathscr{H}_\beta$, through a two steps construction. First we will prove the existence of a convergent cluster expansion (C.E.) in the space time, for every conditional partition function. Secondly we will perform a time's resummation of the truncated functions of the clusters living in the space time, which are projected on the same family of sets of the basis leading to the definition of the effective potentials (Appendixes C, D, E).

2.1. Definitions: Clusters (Fig. 3)

- Two interacting linked loops are incompatible, if there exists at least two loops one in each interacting linked loop, which intersect.
- A cluster $C[T, \vec{\mathscr{A}}^P]$ is defined by two sets:[21, 23] (i) a set of incompatible interacting linked loops $\{\hat{\lambda}_1^I, \ldots, \hat{\lambda}_p^I\} \in \mathscr{CONN}(H; \Sigma_V \mid \Sigma_{\bar{V}})$, (ii) a set of integers $\{\alpha(\hat{\lambda}_1^I), \ldots, \alpha(\hat{\lambda}_p^I)\}$, which are the multiplicities of occurrence of the corresponding interacting linked loops building the cluster, T is the set of jump's times of the constituent loops, and $\vec{\mathscr{A}}^{\mathscr{P}} = \{\vec{A}_1^{\mathscr{P}_1} \cdots \vec{A}_n^{\mathscr{P}_n}\}$ is the family of the pairings corresponding to the linked loops of the cluster. It will be identified with a linked directed graph in the basis.
- $\mathscr{CONN}^T(H; \Sigma_V \mid \Sigma_{\bar{V}}]$ is the set of the clusters built from the loops belonging to $\mathscr{CONN}(H; \Sigma_V \mid \Sigma_{\bar{V}})$.
- $C[T, \vec{\mathscr{A}}^{\mathscr{P}}] \to \mathscr{A}$ is the set of clusters which are projected on the set $\mathscr{A}$.

In the next proposition we express the effective hamiltonian in term of the previous potentials. Notice that, $\Sigma_{\mathscr{A}} \in \Sigma_V \cup \Sigma_{\bar{V}}$, means that $\Sigma_{\mathscr{A}}$ is the restriction to $\mathscr{A}$ of $\Sigma_V \cup \Sigma_{\bar{V}}$.

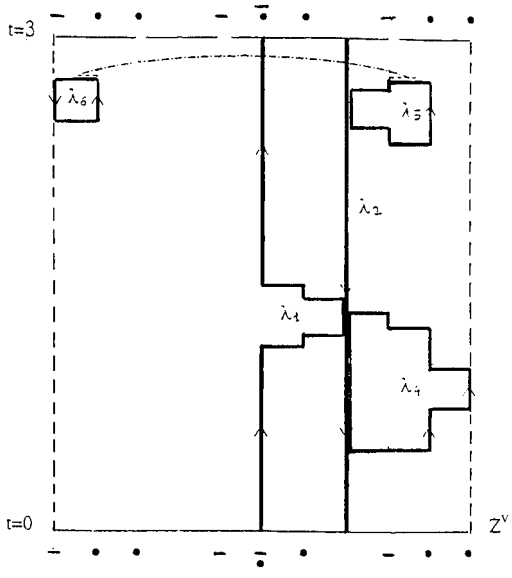

Fig. 3. One cluster built from four intersecting loops: the winding loops λ_1 and λ_2, the nonwinding loop λ_4 and of the linked loop built from the loops λ_5 and λ_6.

2.2. Proposition

The hamiltonian **H** satisfies an S.I. condition (0.11).

(i) Then every conditional partition function is written as:

$$Z\{H; \Sigma_V \mid \Sigma_{\bar{V}}\} = \exp -\beta \left\{ \mathscr{H}^o_\infty[\Sigma_V \mid \Sigma_{\bar{V}}) + \sum_{\{\Sigma_{\mathscr{A}} \in \Sigma_{V \cup \bar{V}}\}} \Psi^\beta[\Sigma_{\mathscr{A}}] \right\} \tag{2.1}$$

(ii) Then, for every $\mathscr{A} \in \hat{\mathscr{A}}[H; \Sigma_V \mid \Sigma_{\bar{V}}]$, the effective potentials decay exponentially:

$$\begin{aligned} |\Psi^\beta[\Sigma_{\mathscr{A}}]| &\leqslant \exp -K.T(\mathscr{A}) \\ \Psi^\beta[\Sigma_v] &= \beta^{-1} \log[1 + \exp(-\beta(U - \sigma_v \, |\mu^e|)] \end{aligned} \tag{2.2}$$

Proof. The proof of the proposition is the core of the paper, it is mainly contained in the appendixes. We construct the effective potentials associated to a given hamiltonian $\mathbf{H} \in \mathbf{F}$ in three steps. The first step is the existence of a convergent cluster expansion (C.E.) for the quantum fluctuations leaving in the space time (polymers) , which arise in each conditional partition function $Z\{H; \Sigma_V \mid \Sigma_{\bar{V}}\}$. In fact we will use the discretized

F.K. representation given in the Appendix A. Then, in the Appendixes B and E, we will be able to generalize the method initiated by Dobrushin for the classical case[21] to the quantum case, to prove the existence of a convergent C.E., which requires in our case a first S.I. condition (0.11). Then the truncated functions $\phi(C[T, \mathscr{A}^P])$ are defined as the multiple derivatives of $\ln[Z\{H; \Sigma_V \mid \Sigma_{\bar{V}}\}]$ w.r.t. the discretized s.d. of the loops[21] (see Appendix C for a definition). NOTICE the following crucial fact: every truncated function is defined locally, it depends ONLY of the restriction of the ion's configuration $\{\Sigma_V \mid \Sigma_{\bar{V}}\}$ to the projection of the constituent loops of the clusters on the basis. The second step is a time's resummation of the truncated functions, which are projected onto the same set on the basis. These resummations lead to the definition of the effective potentials for any $\{\Sigma_{\mathscr{A}} \in \mathbf{\Sigma}_{V \cup \bar{V}}\}$:

$$\Psi^{\beta}[\Sigma_{\mathscr{A}}] = \sum_{\{C[T, \mathscr{A}^P] \to \mathscr{A}\}} \phi(C[T, \mathscr{A}^P]) \tag{2.3}$$

The potentials are shown to decay exponentially in the tree distance in the Propositions C.2.2. and E.3.1.

Notice that the potentials depend of β, but that the upper bounds on the tail's potentials do not, except the weights of the winding loops of $\mathscr{L}^1_o$, which contribute to the chemical potentials, they are infinite for $\beta = 0$. This is harmless because we can substract the constant β^{-1} per lattice site for small β. ▌

2.3. Corollary

The hypothesis of 1.2.2. holds.

(i) The hamiltonian $\mathbf{H}$ admits an effective hamiltonian $\mathscr{H}_{\beta}$ defined by:

$$\mathscr{H}_{\beta}[\Sigma_V \mid \Sigma_{\bar{V}}] = \mathscr{H}^{o}_{\beta}[\Sigma_V \mid \Sigma_{\bar{V}}] + \cdots + \mathscr{H}^{p}_{\beta}[\Sigma_V \mid \Sigma_{\bar{V}}] + \sum_{\{\Sigma_{\mathscr{A}} \in \Sigma_{V \cup \bar{V}} \mid |\mathscr{A}| > p+1\}} \Psi^{\beta}[\Sigma_{\mathscr{A}}] \tag{2.4}$$

where the p order truncated hamiltonian is defined by:

$$\mathscr{H}^{p}_{\beta}[\Sigma_V \mid \Sigma_{\bar{V}}] =: \sum_{\{\Sigma_{\mathscr{A}} \in \Sigma_{V \cup \bar{V}} \mid |\mathscr{A}| = p\}} \Psi^{\beta}_{p}[\Sigma_{\mathscr{A}}]$$

$|\mathscr{A}| = p$ means that the clusters which build the potentials contain p jumps.

(ii) There exists a strictly positive constant A_p such that: if $\beta \frac{\|T\|^{p+1}}{[\tilde{U}-|\tilde{\mu}^c|]^p} > A_p$, the hamiltonian $\mathscr{H}_\beta$ admits a p order LT decomposition:

$$\mathscr{H}_\beta[\Sigma_V \mid \Sigma_{\bar{V}}] = \mathscr{H}^o_\infty[\Sigma_V \mid \Sigma_{\bar{V}}] + \cdots + \mathscr{H}^p_\infty[\Sigma_V \mid \Sigma_{\bar{V}}] + R^{\{\beta, w\}}_p[\Sigma_V \parallel \Sigma_{\bar{V}}] + \sum_{\{\Sigma_\mathscr{A} \in \Sigma_{V \cup \bar{V}} \mid |\mathscr{A}| > p+1\}} \Psi^\beta[\Sigma_\mathscr{A}] \tag{2.5}$$

where

$$\mathscr{H}^p_\beta[\Sigma_V \mid \Sigma_{\bar{V}}] =: \mathscr{H}^p_\infty[\Sigma_V \mid \Sigma_{\bar{V}}] + \mathscr{H}^p_{\{\beta,\omega\}}[\Sigma_V \mid \Sigma_{\bar{V}}]$$

$$R^{\{\beta, w\}}_p[\Sigma_V \parallel \Sigma_{\bar{V}}] =: \sum_{\{\Sigma_\mathscr{A} \in \Sigma_{V \cup \bar{V}} \mid |\mathscr{A}| \leqslant p\}} \Psi^{\{\beta,\omega\}}[\Sigma_\mathscr{A}]$$

Proof. Part (i): We use the Proposition 2.2. to define the effective hamiltonian. Part (ii): We decompose the p order effective potentials into two parts:

$$\Psi^\beta_p[\Sigma_\mathscr{A}] =: \Psi^{\{\beta,\omega\}}_p[\Sigma_\mathscr{A}] + \Psi^\infty_p[\Sigma_\mathscr{A}] \quad \blacksquare$$

The Proof of the Theorem. The finite volume correlation functions restricted to the sub-algebra $\mathscr{S}$, within the set of the b.c. $\Sigma_{\bar{V}}$, coincide with the finite volume Gibbs expectations built from $\mathscr{H}_\beta \in \mathbf{I}$.

$$\langle \sigma_X \rangle^{\mathbf{H}}(\Sigma_{\bar{V}}) = E^{\mathscr{H}_\beta}[\sigma_X \mid \Sigma_{\bar{V}}] \tag{2.6}$$

We know from the Proposition C.2.2. that the tail's potentials decay exponentially, and that, at low temperature, the potentials contained in $\mathscr{H}^q_{\{w,\beta\}}$, for $q \leqslant p$ are of order $p+1$. Then the infinite volume limits of the correlation functions, restricted to $\mathscr{S}$, within the set of the boundary conditions $\Sigma_{\bar{V}}$, are the solutions of the D.L.R. equations of the Gibbs fields defined from the hamiltonian $\mathscr{H}_\beta$. Secondly we prove that the effective hamiltonian $\mathscr{H}_\beta$ is translation invariant. As $\mathbf{H}$ is translation invariant, $\mathscr{H}^o_\beta$ is translation invariant. The effective potentials are built from the translation invariant hoppings, they are translation invariant. $\blacksquare$

3. PIROGOV–SINAI THEORY: PHASE DIAGRAM

We treat the two dimensional case. The extension to higher dimensions is easy. We deduce from the theorem that, every hamiltonian $\mathbf{H} \in \mathbf{F}$ satisfying an S.I. condition (0.11), admits an effective hamiltonian $\mathscr{H}_\beta \in \mathbf{I}$. Then we get the phase diagram of $\mathbf{H} \in \mathbf{F}$ by applying to the effective hamiltonian

$\mathscr{H}_\beta$ the classical P.S. theory extended to infinite range many body potentials performed in refs. 16a and b, we will use the optimal results of ref. 17 valid for potentials decaying exponentially. The main feature of our approach is the existence of a hierarchy of phase diagrams via the "p order P.S. theory" built from the L.T. p order decomposition of the effective hamiltonian $\mathscr{H}_\beta$. An application to the FK model is given in ref. 5.

3.1. Definitions

• The p order P.S. theory for the hamiltonian $\mathbf{H} \in \mathbf{F}$ has a twofold meaning:

(1) the P.S. theory applies to the truncated effective hamiltonian $\mathscr{H}_\infty^{\leqslant p}$,

(2) the hamiltonians $\mathscr{H}_\beta^{>p}$ and $\mathscr{H}_{\{w,\beta\}}^{\leqslant p}$ act as perturbations of the truncated hamiltonian $\mathscr{H}_\infty^{\leqslant p}$.

• A lattice block of size $n \times n$ is such that each square is contained in n^2 blocks. A $n \times n$ block has to contain all the interactions of the hamiltonian strictly contained in the block.

• A good block is a block with a configuration which minimizes the energy. A block which is not a good block is called a bad block.

• A local ground state is a connected set of good blocks.

• The family of the ground states of the type $[r, q]$ is denoted by $\hat{\Sigma}^{[r,q]} = \{\Sigma_1^{[r,q]}, \ldots, \Sigma_n^{[r,q]}\}$, they are indexed by the order r in the hierarchy, and by the class number q in this order.

• A contour is a pair $\{\Theta, \gamma_\Theta\}$, where Θ is a maximal connected set of blocks, and γ_Θ is a configuration of bad blocks defined on Θ. The inner and the outer boundaries of the contours have to be specified, they will be singled out when it is needed. We use the same symbol for the contour and for its support.

• A configuration in V is represented by a family of contours $\Gamma^{[r,q]} = \{\gamma_1^{[r,q]}, \ldots, \gamma_n^{[r,q]}\}$. In general the set of contours contains open contours, which are connected to the boundary of V. In the following, we consider the b.c. defined in $\bar{V}$ by good blocks of the same ground state. Then a configuration in V is defined by a set of closed contours. Next we define, for each configuration Σ_V, the subset of the exterior contours $\Gamma_{ext(q)}^{[r,q]} =: \{\gamma_1^{[r,q]}, \ldots, \gamma_s^{[r,q]}\}$ and the subset of the interior contours $\Gamma_{int.}^{[r,q]} = \{\gamma_{s+1}^{[r,q]}, \ldots, \gamma_n^{[r,q]}\}$ contained in $\{\bigcup_{i=1}^{i=r} Int(\gamma_i^{[r,q]}\}$.

Next we construct the hierarchy of the phase diagrams for the hamiltonians belonging to **F** and satisfying an S.I. condition. We begin by the zeroth order.

3.2. The Zero Order P.S. Theory: Phase Diagram of $H \in F$ Obtained at the Zero Order: Case I

We start from the zero order decomposition of the effective hamiltonian $\mathscr{H}_\beta$.

$$\mathscr{H}_\beta(V) = \mathscr{H}^o_\infty(V) + \mathscr{H}^{>o}_\beta(V) + \mathscr{H}^o_{\{w,\beta\}}(V) \tag{3.1}$$

3.2.1. The P.S. Theory for $\mathscr{H}^o_\infty$

We suppose that the manifold $\mathscr{D}$ is partitioned into a family of $|M_o|$ submanifolds $\{\mathscr{D}^{o,a_o}\}_{a_o \in M_o} \subset \mathscr{D}$, in which the P.S. theory can be applied to the truncated hamiltonian $\mathscr{H}^o_\infty$. Notice that this family contains is not empty, because for the large values of the chemical potentials $|\mu^e - \mu^i|$, the P.S. can be applied. The requirements of the P.S. theory are twofold.

(i) There exists $|M_o|$ P.S. decompositions of $\mathscr{H}^o_\infty$.

$$\mathscr{H}^o_\infty(V) = \mathscr{H}^{\{[o,a_o],o\}}_\infty(V) + \sum_{i=1}^{i=n} \epsilon_i^{[o,a_o]} \mathscr{H}^{\{[o,a_o],i\}}(V) \tag{3.2}$$

$\epsilon_i^{[o,a_o]}$ are real functions of the coupling constants of $\mathscr{H}^o_\infty$. The hamiltonian $\mathscr{H}^{\{[o,a_o],o\}}_\infty$ has a finite set of ground states built with $L_o^{a_o} \times L_o^{a_o}$ good blocks: $\hat{\Sigma}^{[o,a_o]} \equiv \{\hat{\Sigma}_1^{[o,a_o]}, \ldots, \Sigma^{[o,a_o]}_{n_{[o,a_o]}}\}$, moreother the hamiltonians $\mathscr{H}^{\{[o,a_o],i\}}$ have to remove the degeneracies of the ground states of $\hat{\Sigma}^{[0,a_o]}$.[15] The b.c. are defined by patching $\bar{V}$ by $L_o^{a_o} \times L_o^{a_o}$ good blocks of the same ground state. A configuration Σ_V, compatible with the b.c., is described by a family of closed contours. $\Gamma^{[o,a_o]} = \{\gamma_1^{[o,a_o]}, \ldots, \gamma_n^{[o,a_o]}\}$ including their inner and outer configurations.

(ii) The Peierls condition holds for the contours of $\Gamma^{[o,a_o]}$ built from $\mathscr{H}^o_\infty$: there is a strictly positive function $C_o^{[o,a_o]}$ depending of $J_{A,B}$ and of μ^e, (in fact this condition has to be satisfied for every kind of contours), such that the energy w.r.t. $\mathscr{H}^o_\infty$ of a given contour is bounded by:

$$|\eta^o_\infty(\gamma_i^{[o,a_o]})| > C_o^{[o,a_o]} \times |\gamma_i^{[o,a_o]}| \tag{3.3}$$

3.2.2. The Zero Order P.S. Theory for $\mathscr{H}_\beta$

The P.S. theory, which is valid for $\mathscr{H}^o_\infty$ will be extended to the full hamiltonian $\mathscr{H}_\beta$, if we verify the three following conditions.

(i) The existence of $|M_o|$ P.S. decompositions of $\mathscr{H}_\beta$, which are the P.S. decompositions of $\mathscr{H}^o_\infty$ modified by the terms of higher order.

(ii) The Peierls condition for $\mathscr{H}_\beta$: there are two strictly positive functions $C_o^{[o,a_o]}$ and $C_1^{[o,a_o]}$ depending of $J_{A,B}$ and of μ^e such that:

$$|\eta^o_\beta(\gamma_i^{[o,a_o]})| > \left[C_o^{[o,a_o]} - C_1^{[o,a_o]} \frac{\|T\|^2}{\tilde{U} - |\tilde{\mu}^e| - \|T\|} \right] \times |\gamma_i^{[o,a_o]}| \tag{3.4}$$

The additional term include the upper bound on the higher order potentials including the contribution of the winding loops, which supports are the supports of the potentials of $\mathscr{H}^o_\infty$, then we use the estimate (b) of the Proposition C.2. to get the estimate.

(iii) The tail potentials decay exponentially, which follows from the Proposition C.2.

Finally, following ref. 17, we deduce that the zeroth order phase diagram of $\mathbf{H} \in \mathbf{F}$ is a smooth deformation of the phase diagrams of $\mathscr{H}^o_\infty$ in eachone of the shrinked domain $\{S_0(\mathscr{D}^{[o,a_o]})\}_{a_o \in M_o}$.

Note. The phase diagram of $\mathscr{H}_\beta$ remains unknown in $\mathscr{D} - \{\bigcup_{a_o=1}^{a_o=|M_o|} S_0(\mathscr{D}^{[o,a_o]})\}$. Next we refine the zeroth order phase diagram by going to the next order decomposition of $\mathscr{H}_\beta$, in general two cases will appear at the first order.

- Case I. The same phase transitions appear in the domains $S_0(\mathscr{D}^{[o,a_o]})$, which are slightly enlarged meanwhile the range of temperature slightly increases.
- Case II. The most interesting case is the appearance of new domains emerging in between the hypersurfaces for which, there is infinite degeneracy of the ground states for the hamiltonian $\mathscr{H}^o_\infty$. Such a situation occurs when the truncated hamiltonian $\mathscr{H}^1_\infty$ lift the degeneracy of the ground states, then new domains appear in which new quantum phase transitions appear.

As the structure is general, we will suppose that the phase diagram has been built up to the order $p-1$, then we build the p order.

3.3. The *p* Order P.S. Theory: The Phase Diagram of H ∈ F Obtained at the Order *p*

We start from the p order decomposition of $\mathscr{H}_\beta$:

$$\mathscr{H}_\beta(V) = \mathscr{H}^{\leqslant p}_\infty(V) + \mathscr{H}^{>p}_\beta(V) + \mathscr{H}^{\leqslant p}_{\{w,\beta\}}(V) \tag{3.5}$$

We suppose that the manifold $\mathscr{D}$ is partitioned into a family of $|M_p|$ submanifolds $\{\mathscr{D}^{[p,a_p]}\}_{a_p \in A_p} \subset \mathscr{D}$, in which the P.S. theory can be applied to the truncated hamiltonian $\mathscr{H}_\infty^{\leqslant p}$. This family contains at least the family of the shrinked domains $S_{p-1}(\mathscr{D}^{[p-1,a_{p-1}]})$ constructed at the order $p-1$.

3.3.1. The P.S. Theory for the Truncated Effective Hamiltonian $\mathscr{H}_\infty^{\leqslant p}$

(i) There exists $|M_p|$ P.S. decompositions of $\mathscr{H}_\infty^{\leqslant p}$.

$$\mathscr{H}_\infty^{\leqslant p}(V) = \mathscr{H}_\infty^{\{[p,a_p],o\}}(V) + \sum_{j=1}^{j=q} \epsilon^*_{[p,a_p],j} \mathscr{H}_\infty^{\{[p,a_p],j\}}(V) \tag{3.6}$$

Notice that the first $|M_{p-1}|$ decompositions at the order p differ from the corresponding decomposition at the order $p-1$ only by additionnal terms of order U^{-p}. The ground states of $\mathscr{H}^{\{[p,a_p],0\}}$ are $\hat{\Sigma}^{[p,a_p]} \equiv \{\Sigma_1^{[p,a_p]},\ldots, \Sigma_{n_{[p,a_p]}}^{[p,a_p]}\}$. A configuration Σ_V compatible with the b.c. defined by good blocks in $\bar{V}$, is represented by a family of closed contours of the type $[p, a_p]$: $\gamma^{[p,a_p]} \equiv \{\gamma_1^{[p,a_p]} \ldots \gamma_n^{[p,a_p]}\}$ including their outer and inner configurations.

Now we have to distinguish between two cases.

Case I. $|M_{p-1}| = |M_p|$. The p order P.S. decomposition is a refinement of a decomposition obtained at lower orders. We follow exactly the same route. We first notice that the size of the corresponding blocks $L_p^{a_p} \times L_p^{a_p}$ defining the ground states generally increases even for the same ground states because the range of the potential generally increases. The Peierls condition is obtained at a higher level of accuracy. Then the domains in which the same phase transitions occur are slightly larger and the range of temperature is slightly larger than those obtained at lower orders.

Case II. $|M_{p-1}| < |M_p|$. There is a new P.S. decomposition: there is a new family of ground states, which can be distinguished at the order p, and which could not be distinguished at the order $(p-1)$. This means, in particular, that the Peierls condition is satisfied for the corresponding contours at the order p.

(ii) The Peierls condition for the contours of the type $[p, a_p]$ w.r.t. $\mathscr{H}_\infty^p$: there exists a strictly positive function $C_o^{[p,a_p]}$ of $J_{A,B}$, of $\{T\}$, and of μ^e such that:

$$|\eta_\infty^{\leqslant p}(\gamma_i^{[p,a_p]} \mid \Gamma_{int.}^{[p,a_p]})| > C_o^{[p,a_p]} \times \frac{\|T\|^{p+1}}{[\tilde{U} - |\tilde{\mu}^e| - \|T\|]^p} |\gamma_i^{[p,a_p]}| \tag{3.7}$$

3.3.2. The p Order P.S. Theory for $\mathscr{H}_\beta$

The p order P.S. theory can be extended from $\mathscr{H}_\infty^{\leqslant p}$ to $\mathscr{H}_\beta$, if we prove the three following conditions.

(i) The P.S. decomposition of $\mathscr{H}_\beta$ is the P.S. decomposition of $\mathscr{H}_\infty^{\leqslant p}$ modified by terms of higher order.

(ii) The Peierls condition for the contours of the type $[p, a_p]$ w.r.t. $\mathscr{H}_\beta$: there exist two strictly positive functions $C_o^{[p, a_p]}$ and $C_1^{[p, a_p]}$ functions of $J_{A, B}$ of $\{T\}$, and of μ^e such that:

$$|\eta_\beta(\gamma_i^{[p, a_p]} \mid \Gamma_{int.}^{[p, a_p]})|$$
$$> \left[C_o^{[p, a_p]} - C_1^{[p, a_p]} \frac{\|T\|}{\tilde{U} - |\tilde{\mu}^e| - \|T\|} \right] \times \frac{\|T\|^{p+1}}{[\tilde{U} - |r(\tilde{\mu}^e)| - \|T\|]^p} |\gamma_i^{[p, a_p]}| \qquad (3.8)$$

The additional term includes the upper bound on the higher order potentials (including the winding loop contribution) of range p, which are estimated in the Proposition C.1.2. estimate b.

(iii) The tail potentials decay exponentially. This follows from the Proposition C.2.

Then, following ref. 17, we deduce that the p order phase diagram of $\mathbf{H}$ is a smooth deformation of the phase diagram of $\mathscr{H}_\infty^{\leqslant p}$ in the shrinked domains $\{S_p(\mathscr{D}^{[p, a_p]})\}_{a_p \in M_p}$.

4. THE HIGH TEMPERATURE REGIME

4.1. Corollary

The hamiltonian $\mathbf{H} \in \mathbf{F}$ satisfies an S.I. condition together with the condition, $(C > 0)$.

$$\beta \left[\frac{\|T\|^2}{|r(\tilde{U})| - \|T\| - |r(\tilde{\mu}^e)|} + |1 - e^{-\beta(|r(\tilde{U})| - |r(\tilde{\mu}^e)| - \|T\|)}| \right] < C \qquad (4.1)$$

(i) There is uniqueness of the correlation functions restricted to the subalgebra $\mathscr{S}$ w.r.t. the set of the boundary conditions included in $\bar{\Sigma}$.

(ii) The correlation functions are analytic functions of the coupling constants.

The proof follows from known results about the classical systems at high temperature.[13]

APPENDIX A. ESTIMATES FOR THE HAMILTONIANS $H \in F$, $J_{X,Y}=0$

We derive the estimates, which are needed to prove the existence of a convergent C.E. for every conditional partition function defined from $\mathbf{H} \in \mathbf{F}$. At first we take $J_{X,Y} = 0$ ($U = \hat{U}$). The dependence of our estimates in $\mathbf{H} \in \mathbf{F}$ will be frequently skipped, because it appear in the notations $\{T, U, \mu^e, \mu^i\}$, where T denotes the set of the hoppings. $r(U)$ (real part of U) can be chosen to be positive because of the symmetries of the hamiltonian. To perform a convergent C.E. for every conditional partition function, we use a lattice approximation for the F.K. representation, which starts from the Trotter formula.[10] Finally we will use the theorems about the limits of sequences of complex analytic functions to go back the continuous case.

A.1. The Discrete Feynman–Kac Representation

The space time $Z^\nu \times \{o, \beta\}$ is converted into a lattice $Z^\nu \times \{0, S\beta\}$, the time's variables take integer values. Let $\Lambda_S = V \times \{0, \beta S\} \subset Z^\nu \times \{0, \beta S\}$ be a subset of the lattice. The main change w.r.t. the continuous case is that the jumps of the loops occur at discrete times. Obviously the geometric and the topological properties of the discrete loops and of the continuous loops are the same. The notations used for the discrete case are used for the continuous case, but supplemented by an additionnal subscript S. We adapt the previous definitions to the discrete case.

- $J(\hat{\lambda})$ is the set of jumps of the non winding loop $\hat{\lambda}_S$. The jumps of a linked loop are supported by a family of pairings $\mathscr{A}^{\mathscr{P}} \equiv \{A_o^{P_o}, \ldots, A_n^{P_n}\}$. The birth's time b_o is a F.K. chosen arbitrarily among the jump's times b_o is the origin of the ordered jump times $\{b_1, \ldots, b_n\}$ counted along the orientation of the loop.

- A winding linked loop as $\hat{\lambda}_S^W \in \mathscr{COND}(H; \Sigma_V \mid \Sigma_{\bar{V}})$ is generally composed of two families of loops: a family of non winding loops $\{\lambda_S^1, \ldots, \lambda_S^p\}$, and a non empty family of winding loops $\{\lambda_S^{w_1}, \ldots, \lambda_S^{w_q}; \lambda_S^{-w_1'}, \ldots, \lambda_S^{-w_r'}\}$, the winding numbers w_j and w_k' are positive integers.

- The winding number of $\hat{\lambda}^W$ is $|W| = \sum_i w_i + \sum_j w_j'$.

- Every winding loop is decomposed into its standard parts and into its bubble parts defined in 1.2.2, the bubble parts contain a set of jumps $J^b(\hat{\lambda}_S^W) = \{J^b(\lambda_S^{w_1}), \ldots, J^b(\lambda_S^{w_q}); J^b(\lambda_S^{-w_1'}), \ldots, J^b(\lambda_S^{-w_r'})\}$, the jumps which connect the bubble parts and the standard part belongs to the standard part. The remaining set of jumps, including the turning jumps is the set:

$J^{st}(\hat{\lambda}^W) = \{J^{st}(\lambda_S^{w_1}),..., J^{st}(\lambda_S^{w_q}); J^{st}(\lambda_S^{-w_1'}),..., J^{st}(\lambda_s^{-w_r'})\}$. The integers m_i are the numbers of the vertical segments of the constituent loops of $\hat{\lambda}_S$, which are in between the times b_{i+1} and b_i. For example, the discrete signed probability (d.s.p.) of the loop $\hat{\lambda}_S$ takes the form:

$$\varphi_S(\hat{\lambda}_S) =: \frac{1}{S^{n+1}} t_{A_o} \times \cdots \times t_{A_n} \prod_{j=1}^{j=p} \epsilon(\lambda_j)$$

$$\times \exp\left\{ -\frac{U}{S} \sum_{i=o}^{i=n-1} m_i(b_{i+1} - b_i) \right\} \chi(b_1,..., b_n) \qquad \text{(A.1)}$$

The d.s.p. of the winding loops are obtained in the same way.

- The discrete conditional partition function associated to the discrete conditional ensemble $\mathscr{CONS}_S(H; \Sigma_V \mid \Sigma_{\bar{V}})$ takes the form:

$$Z_S(H; \Sigma_V \mid \Sigma_{\bar{V}}) = e^{-\beta \mathscr{H}_\infty^o(\Sigma_V \mid \Sigma_{\bar{V}})} \sum_{\{\hat{\lambda}_S^1 \cdots \hat{\lambda}_S^p; \hat{\lambda}_S^{\{w,1\}},..., \hat{\lambda}_S^{\{w,r\}}\} \subset \mathscr{CONS}_S(H; \Sigma_V \mid \Sigma_{\bar{V}})}$$

$$\times \prod_{i=1}^{i=p} \varphi_S(\hat{\lambda}_S^i) \prod_{j=1}^{j=r} \varphi_S(\hat{\lambda}_S^{\{w,j\}}) \qquad \text{(A.2)}$$

The convergence of the discrete partition function to the continuous one is a standard result.[11]

A.1.1. Definitions: Projection of the Loops on the Basis

- The projections of the oriented edges, of the links, of the bond of the loop $\hat{\lambda}_S^I$ on the basis build a linked interacting directed graph with even incidence at each vertex (l.i.d. graph) $\vec{A}^P$ (P is a pairing).

- Two linked interacting loops are equivalent, if their projections on the basis are identical: $\hat{\lambda}_S^I \sim \hat{\mu}_S^I \sim \vec{A}^P$.

- The unoriented graph A^P is the l.i.d. graph $\vec{A}^P$ without its orientation. $\vec{A}^P \to A^P$ is the set of the l.i.d. graphs with the same unoriented graph.

- A complete family of linked winding loops (resp. non winding loops) $\mathscr{F}^w_{[a,a+1)}(\vec{A}^P) \subset \mathscr{CONS}(H; \Sigma_V \mid \Sigma_{\bar{V}})$ (resp. $\mathscr{F}^{n,w}_{[a,a+1)}(\vec{A}^P) \subset \mathscr{CONS}(H; \Sigma_V \mid \Sigma_{\bar{V}})$) is the set of the equivalent linked winding loops (resp. non winding linked loops) projected on $\vec{A}^P$, which birth's time belongs to $[a, a+1)$; the dependence w.r.t. H and $(\Sigma_V \mid \Sigma_{\bar{V}})$ is skipped, because it is contained in the structure of $\vec{A}^P$ itself. We define four sums:

$$\Phi^{\{p,\,nw\}}_{S,\,\{x,\,b\}}(\Sigma_V \mid \Sigma_{\bar V}) = \sum_{\{\hat\lambda_S \in \mathscr{COND}(H;\,\Sigma_V \mid \Sigma_{\bar V}) \mid \{x,\,b\} \in \hat\lambda_S\}} S \times \varphi_S(\hat\lambda_S) \times \left[\frac{b_1}{S}\right]^p$$

$$\Phi^{\{p,\,w\}}_{S,\,\{x,\,b\}}(\Sigma_V \mid \Sigma_{\bar V}) = \frac{\exp -\beta\{r(U) - \sigma_x\,|r(\mu_e)|\}}{\beta}$$

$$+ \sum_{\{\hat\lambda^W_S \in \mathscr{COND}(\Sigma_V \mid \{x,\,b\} \in \hat\lambda^W_S\}} S \times \varphi_S(\hat\lambda^W_S) \times \left[\frac{b_1}{S}\right]^p$$

$$\Phi^p_S[\mathscr{F}^{nw}_{[a,\,a+1)}(\vec A^P_i)] = \sum_{\{\hat\lambda_S \sim \vec g_i) \mid \{x,\,b\} \in \hat\lambda^{nw}_S;\,b \in [a,\,a+1)\}} S \times \varphi_S(\hat\lambda^{nw}_S) \times \left[\frac{b_1}{S}\right]^p$$

$$\Phi^p_S[\mathscr{F}_{[a,\,a+1)}(\vec A^P_i)] = \frac{\exp -\beta\{r(U) - \sigma_x\,|r(\mu_e)|\}}{\beta}$$

$$+ \sum_{\{\hat\lambda_S \sim \vec A^P_i \mid \{x,\,b\} \in \hat\lambda_S;\,b \in [a,\,a+1)\}} S \times \varphi_S(\hat\lambda_S) \times \left[\frac{b_1}{S}\right]^p$$

- $\Phi^p_S[U, T]$ (resp. $\Phi^p_S[U, T, \mu_e]$) is the uniform upper bound for $\Phi^p_{S,\,\{x,\,b\}}(\Sigma_V \mid \Sigma_{\bar V})$ (resp. $\Phi^p_{S,\,\{x,\,b_o\}}(\Sigma_V \mid \Sigma_{\bar V})$) w.r.t. the underlying ion's configuration, and w.r.t. their birth's time. The corresponding limits $S \to \infty$, when it exists, is denoted by $\Phi^p[U, T]$ (resp. by $\Phi^p[U, T, \mu_e]$).

- $\Phi[\mathscr{F}_{[a,\,a+1)}(\vec A^P)]$ (resp. $\Phi[\mathscr{F}^{nw}_{[a,\,a+1)}(\vec A^P)]$) is the limit $S \to \infty$ (if it exists) of $\Phi^p[\mathscr{F}^S_{[a,\,a+1)}(\vec A^P)]$ and respectively $\Phi[\mathscr{F}^{S,\,nw}_{[a,\,a+1)}(\vec A^P)]$. In the following we will rather consider $\Phi[\mathscr{F}_{[o,\,1)}(\vec A^P)]$, (in short notation $\Phi^{p,\,nw}[\vec A^P]$), and respectively $\Phi[\mathscr{F}^S_{[o,\,1)}(\vec A^P)]$, (in short notation $\Phi^{p,\,w}[\vec A^P]$), because our bounds will be time invariant.

Note. In this appendix, we have to find upper bounds for various sums of d.s.p. It is easier to replace the Riemann sums by the corresponding integrals. This is a consequence of the two following facts.

- First we bound the discrete sums by their modulus.
- Secondly the Riemann sums are bounded from above by the corresponding integral, because the integrand is a decreasing function and the discrete sum begins from one.

Proposition A.2. The hamiltonians $\mathbf{H} \in \mathbf{F}$; $\mathbf{J}_{X,\,Y} = 0$ satisfy an S.I. condition (0.11).

(a) Then the following estimates hold for every directed graph $\vec A^P \neq v$, and for every β.

$$\text{(i)} \quad |\Phi^p[\mathscr{F}^{nw}_{[o,1)}(\vec{A}^P)]| \leqslant \frac{p!}{r(U)^p} \times \frac{\prod_{i=o}^{i=n} |t_{A_i}|}{r(U)^n}$$

$$\text{(ii)} \quad |\Phi^p[\mathscr{F}^{w}_{[o,1)}(\vec{A}^P)]| \leqslant \left\{\frac{p!}{[r(U)-r(\mu^e)]^p}\right\} \prod_{i=o}^{i=n} |2^{\frac{A_i}{2}} t_{A_i}| \left\{\frac{1}{r(\widetilde{U})-|r(\widetilde{\mu}_e)|}\right\}^n$$

$$\text{(iii)} \quad |\Phi^p[U,T]| \leqslant \left\{\frac{p!}{r(U)^p}\right\} \times \frac{\|T\|^2}{r(U)-\|T\|} \tag{A.3}$$

$$\text{(iv)} \quad |\Phi^p\{U,T,\mu^e\}| \leqslant \beta^{-1} \exp(-\beta[r(U)-|r(\mu^e)|) + \left\{\frac{p!}{[r(U)-r(\mu^e)]^p}\right\} \times \frac{\|T\|^2}{r(U)-|r(\mu_e)|-\|T\|}$$

(b) There exists two positive constants C_p and A_p such that, if $\beta \frac{\|T\|_\infty^{p+1}}{r(U)^p} > A_p$, then:

$$|\Phi[U,T,\mu_e]| \leqslant C_p \frac{\|T\|^{p+1}}{[r(\widetilde{U})-r(\widetilde{\mu}^e)]^p} \tag{A.4}$$

Proof of Estimate a(i). We begin with a complete family of non winding loop $\hat{\lambda}_S$, which projection on the basis is the directed graph $\vec{A}^P$. The jumps' intensities of these loops are $\{t_{\{x_o,y_o\}} \cdots t_{\{x_n,y_n\}}\}$, the birth's time is b_o, the other jump's times of the loops are $\{b_1,\ldots,b_n\}$. We bound the Riemann sums by the integral.

$$\left| \prod_{i=o}^{i=n} t_{\{x_i,y_i\}} \left[\sum_{b_o=o}^{b_o=S} \frac{1}{S} \sum_{b_n=n}^{b_n=\infty} \frac{1}{S} e^{-U\frac{b_n}{S}} \sum_{b_{n-1}=n-1}^{b_{n-1}=b_n} \frac{1}{S} \cdots \sum_{b_1=1}^{b_1=b_2} \frac{b_1^p}{S^{p+1}} \right] \right|$$
$$\leqslant \prod_{i=o}^{i=n} |t_{\{x_i,y_i\}}| \left[\int_0^1 ds_o \int_0^\infty e^{-s_n r(U)}\, ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} s_1^p\, ds_1 \right]$$

Next, by an easy computation, we get the upper bound:

$$|\Phi_S^{p,nw}[\vec{A}^P]| \leqslant \prod_{i=o}^{i=n} |t_{\{x_i,y_i\}}| \left[\int_0^1 ds_o \int_0^\infty e^{-s_n r(U)}\, ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} s_1^p\, ds_1 \right]$$
$$\leqslant p! \times \prod_{i=1}^{i=n} |t_{\{x_i,y_i\}}| \int_o^\infty e^{-s_n r(U)} \times \frac{s_n^{\{n+p\}}}{(n+p)!} ds_n$$
$$\leqslant \frac{p!}{r(U)^p} \times \frac{\prod_{i=o}^{i=n} |t_{\{x_i,y_i\}}|}{r(U)^n} \tag{A.5}$$

We have done three successive integrations.

- The integration over the birth's time s_o in the interval $(0, 1]$
- The integrations of the ordered jumps' times $\{s_1 \cdots s_n\}$.
- $n+p$ integrations by part.

The extension of the previous computation to the case of a linked non winding loops is straightforward. Let $\{\lambda_1,..., \lambda_p\}$ be the constituent loops of the linked loop $\hat{\lambda}$.

• A jump's time, which is integrated in the s.p. of a loop λ_i, is said to be an active time for λ_i, if not, it is said to be a dummy time for λ_i. We choose a jump's time of λ_1 as birth's time, then the jumps' times of λ_1 are taken to be active except if simultaneous jumps occur in λ_1 itself, due to the fermionic potential, in this case we choose arbitrarily one such jump's time to be active. Then we order the active jumps' times of λ_1, taking the birth's time as the origin, according to the orientation of the loop. The loop λ_1 is linked to at least one loop, say λ_2 by, at least, one simultaneous jump. The common jump's times between λ_1 and λ_2 are considered as active times for λ_1, they are dummy for λ_2. Next we proceed for λ_2, we order the active jump's times of λ_2 as we did for λ_1, taking as birth time for λ_2 a dummy time of λ_2. We iterate this process up to the last loop λ_p. Now, to get the estimate for the linked loop $\hat{\lambda}$, we begin the integration by the active jump's times of λ_p (if there are no, we consider λ_{p-1}, and so on) by using the previous estimates for the non linked loops. The integration of the active times is iterated up to the loop λ_1. In fact we get the products of the integrated s.d. of each constituent loop over their the active times. The final integration of the birth's time of λ_1 over $[o, 1)$ gives 1. ▌

Proof of Estimate a(ii). We have two cases to consider. First we suppose that s_1^p belongs to the bubble part of the loop. A winding linked loops $\hat{\lambda}^W$, which is projected on the directed graph $\vec{A}^P \neq v$, is built from a family of p non winding loops $(\lambda_1,..., \lambda_p)$ and from a family of r linked winding loops $(\lambda_1^{w_1},..., \lambda_q^{w_q}; \lambda_{q+1}^{-w'_{q+1}},..., \lambda_r^{-w'_r})$. The s.p.d. of $\hat{\lambda}^W$ is bounded by:

$$|\varphi(\hat{\lambda}^W)| \leqslant |t_{A^1}| \times \cdots \times |T_{A^n}| \times \chi(s_1,..., s_n) \times \exp\left\{-r(U) \sum_{i=o}^{i=n-1} m_i(s_{i+1} - s_i)\right\}$$
$$\times \exp -\beta w_k \{r(U) + r(\mu^e)\} \times \exp -\beta w'_l \{r(U) - r(\mu^e)\} \qquad \text{(A.6)}$$

We need an S.I. condition (0.11) to be able to perform the following operations.

- We decompose the jump's times of $\hat{\lambda}^W$ into the active times and into the dummy times, as we did previously.
- We decompose each winding linked loops into their standard parts and into their bubble parts (Figs. 3a and b).
- We factorize the hoppings of the standard parts contained in $J^{st}(\hat{\lambda}^W)$, we recall that the first jump (turning jump) of every bubble part is included in $J^{st}(\hat{\lambda}^W)$).
- We integrate the r active ordered jumps' times of $J^b(\hat{\lambda}^W)$.
- We factorize the product of the hoppings of the jumps contained in $J^b(\hat{\lambda}^W)$, and in the non winding linked loops.
- We integrate over the $(n-r)$ active jump's times of $J^b(\hat{\lambda}^W)$ (excluding the turning jumps) by using the estimate i), notice that the bubble parts of the loop have the same structure as the non winding linked loops.
- We integrate over the active jump's times of the non winding linked loops.

$$
\begin{aligned}
|\Phi^{p,w}[\vec{A}^P]| &\leqslant \frac{p!}{r(U)^p} \exp -\beta\,|W|\,\{r(U)-|r(\mu_e)|\} \times \left\{ \prod_{j \in J^b(\hat{\lambda}^W)} \frac{|t_{A_j}|}{r(U)} \right\} \\
&\quad \times \left\{ \prod_{i \in J^{st}(\hat{\lambda}^W)} |t_{A_i}| \right\} \times \left\{ \int_0^{\beta|W|} ds_p \cdots \int_o^{s_1} ds_1 \right\} \\
&\leqslant \frac{p!}{r(U)^p} \exp -\beta\,|W|\,\{r(U)-|r(\mu_e)|\} \times \left\{ \prod_{i \in J^{st}(\hat{\lambda}^W)} |t_{A_i}| \right\} \\
&\quad \times \frac{[\beta\,|W|]^{|J^{st}(\hat{\lambda}^W)|}}{|J^{st}(\hat{\lambda}^W)|!} \times \left\{ \prod_{j \in J^b(\hat{\lambda}^W)} \frac{|t_{A_j}|}{r(U)} \right\} \qquad \text{(A.7a)}
\end{aligned}
$$

We recall that s_1^p belongs to the standard part of the loop, then we get:

$$
\begin{aligned}
|\Phi^{p,w}[\vec{A}^P]| &\leqslant p!\{\beta\,|W|\}^p \exp -\beta\,|W|\,\{r(U)-|r(\mu_e)|\} \\
&\quad \times \left\{ \prod_{i \in J^{st}(\hat{\lambda}^W)} |t_{A_i}| \right\} \times \frac{[\beta\,|W|]^{|J^{st}(\hat{\lambda}^W)|}}{|J^{st}(\hat{\lambda}^W)|!} \times \left\{ \prod_{j \in J^b(\hat{\lambda}^W)} \frac{|t_{A_j}|}{r(U)} \right\} \qquad \text{(A.7b)}
\end{aligned}
$$

Next we sum over all the possible decompositions of the linked loop into non winding parts and winding parts, such that their union is projected onto the directed linked graph $\vec{g}$. We notice that the action of a fermionic potential with hopping t_{A_i}; creates $\frac{A_i}{2}$ jumps, which involve either winding loops or non winding loops, the number of choices is obviously

bounded by: $\sum_p C^p_{\frac{A_i}{2}} = 2^{\frac{A_i}{2}}$. More other each jump appear either in a bubble of a non winding loop or in the standard part. This means that each directed graph contained in $\vec{A}^P$ can be patched with oriented segments, which are either the projections of standard parts or of bubble parts of a loop. Let n be the length of the directed graph, there are C^q_n decompositions of a directed graph into q disjoint oriented segments, because we need q points to build q disjoint segments on a circle. Finally we sum over the winding numbers. Notice that if $W > 0$, $q > 1$, and that $|W| \leqslant q$.

$$|\Phi^{p,w}[\vec{A}^P]| \leqslant p! \times \sum_{|W|=0}^{|W|=\infty} \left[\frac{1}{r(U)^p} + \{\beta\,|W|\}^p \right] \times \prod_{i=o}^{i=n} |T_{A_i}| \times 2^{\frac{A_i}{2}}$$

$$\sum_{q=1}^{q=n} C^q_n \times \exp -\beta\,|W|\,\{r(U) - |r(\mu_e)|\} \times \frac{[\beta\,|W|]^q}{q!} \times \frac{1}{r(U)^{(n-q)}}$$

$$\leqslant p! \prod_{i=o}^{i=n} |t_{A_i}| \times 2^{\frac{A_i}{2}} \sum_{q=1}^{q=n} C^q_n \left[\left\{ \frac{1}{r(U) - |r(\mu^e)|} \right\}^{q+p} \left\{ \frac{1}{r(U)} \right\}^{\{n-q\}} \right.$$

$$\left. + \left\{ \frac{1}{r(U) - |r(\mu^e)|} \right\}^{q} \left\{ \frac{1}{r(U)} \right\}^{\{n-q+p\}} \right]$$

$$\leqslant \frac{p!}{[r(U) - r(\mu^e)]^p} \prod_{i=o}^{i=n} |t_{A_i}| \times 2^{\frac{A_i}{2}} \times \left[\frac{1}{r(U) - |r(\mu_e)|} \right]^n \qquad \text{(A.8)}$$

The second inequality follows from the following inequality valid for $(r(U) - |r(\mu_e)| > 0$ and $p > 0$:

$$\sum_{W=1}^{W=\infty} e^{-\{\beta\,|W| \cdot (r(U) - |r(\mu_e)|)\}} \times \frac{\{\beta \cdot |W|\}^p}{p!} < \left\{ \frac{1}{r(U) - |r(\mu_e)|} \right\}^p \qquad \text{(A.9)}$$

To get the proof of this inequality, we just bound the sum by the integral. ▌

Proof of Estimate a(iii). We need first an auxiliary lemma, which requires two definitions. ▌

Definitions.

- $I^{(n)}_{(x_0,\,t_o)}(H[\Sigma_V \mid \Sigma_{\bar{V}})$ is the sum of the hoppings of the loops, which can exist through the action of the hamiltonian $\mathbf{H} \in \mathbf{F}$ starting from the point (x_o, t_o) after n (simultaneous) jumps.

• $N_n(T, U)$ is the uniform bound for $I^{(n)}_{(x_0, t_o)}(H[\Sigma_V \mid \Sigma_{\bar{V}}])$ w.r.t. the configuration $[\Sigma_V \mid \Sigma_{\bar{V}}]$, and w.r.t. the birth's time. Next we define:

$$|T|_n = \sum_{\{0 \in A \subset Z^\nu \mid |A| \leqslant n\}} \left[\frac{A}{2}\right]! \times |T_A|$$

Then the limit: $\lim_{\{n \to \infty\}} |T_n| = |T|$ exists because $\|T\| < \infty$.

Lemma A.2.1. For any hamiltonian $\mathbf{H} \in \mathbf{F}$, the following bound holds:

$$N_n(T, U) \leqslant |T|^n \tag{A.10}$$

Proof. The estimate is obtained through a recurrent procedure on the number of successive jumps (level).

Level 1. The first action of the fermionic hamiltonian H^f.

We start from the point $\{a_o, s_o\}$, at which either a jump or a family of simultaneous jumps occurs under the action of the Hamiltonian H^f. One of the following situations occurs:

• One monomial of the interaction acts, then the electron, which is at the point $\{a_o, s_o\}$ jumps to the point $\{b_o, s_o\}$, with the corresponding hopping t_{a_o, b_o}. To take into account all the possible jumps, we sum over all the different hoppings containing one point. We get the upper bound: $|T|_1$.

• Next we consider the action of a product of two monomials. There exists two different pairings between the creation and the annihilation operators of the potential. A given pairing creates two simultaneous jumps for two electrons which jump from the points (a_o, s_o) and (a_1, s_o) up to the points (b_o, s_o) and respectively to (b_1, s_o), the corresponding "hopping" is $t_{(a_o, a_1)(b_o, b_1)}$. Let us remark first that the action of the fermionic interaction can be either in the same loop or in two different linked loops, and secondly that there are two possible pairings. Then we get the upper bound for the sum of the fermionic intensities containing one point: $2! \sum_{\{a_1, b_o, b_1\}} |t_{(a_o, a_1; b_o, b_1)}|$

Next we consider the case of a fermionic interaction, which is composed of a product of p monomials, then we can build $p!$ different pairings, and, at most, $p!$ loops. Then, after the first step, we have the upper bound:

$$N_1(T,U) \leqslant |T|_1 + 4 \sum_{\{b_o, a_1, b_1\}} |t_{(a_o, a_1; b_o, b_1)}|$$

$$+ \cdots + n! \sum_{\{a_1,\ldots,a_{n-1}; b_o,\ldots,b_{n-1}\}} |t_{(a_o,\ldots,a_{n-1}; b_o,\ldots,b_{n-1})}|$$

$$\leqslant |T|_n \tag{A.11}$$

Level m. The mth action of the fermionic hamiltonian H^f.

We iterate the process defined above up to the time s_{m-1}. We consider the set of points which are the intersection of the hyperplane $s = s_m$ and of the branches of the constituent loops of the linked loop, which were created at previous times $s_o \cdots s_{m-1}$. Then, at the time $s_m > s_{m-1}$ the fermionic interaction acts, this means that, at each space time point attained at the previous time, the process defined at the first step, is repeated. The following upper bound follows:

$$N_m(T,U) \leqslant N_{(m-1)}(T,U) \times N_1(T,U) \leqslant |T|_n^{m+1} \tag{A.12}$$

The case of infinite monomials interaction can be considered as well, provided that the "hoppings" satisfy the condition (A.10): $|T| < \infty$, to get

$$N_m(T,U) \leqslant |T|^{m+1}$$

Next we derive the estimate a(iii). We point out that the upper bound obtained in a(i), either for a linked non winding loop with m jumps, differs by the hoppings only. So that, to sum over the different loops, i.e., over the different products of m hoppings, we use the estimate of the Lemma A.2.1: we replace every hopping contained in the estimate (i) by $|T|$ and then by $\|T\|$ to get.

$$|\Phi^p\{U,T\}| \leqslant p! \times \sum_{m=1}^{\infty} \frac{|T|^{m+1}}{r(U)^{m+p}} \leqslant \frac{p!}{r(U)^p} \times \frac{\|T\|^2}{r(U) - \|T\|} \tag{A.13}$$

Proof of Estimate a(iv). We start from the estimate a(ii), then we performed the following successive summations.

- First we start from the Lemma A.2.4., in the estimate, we replace each hopping by $\|T\|$.
- Next we sum over the number of jumps.

- Finally we add the contribution of the winding loops without jump.

$$
\begin{aligned}
|\Phi^p[U,T,\mu_e]| &\leqslant \left\{\frac{p!}{[r(U)-r(\mu^e)]^p}\right\}\|T\|\sum_{n=1}^{n=\infty}\left[\frac{\|T\|}{r(U)-|r(\mu_e)|}\right]^n \\
&\quad+\beta^{-1}\exp(-\beta[r(U)-|r(\mu^e)|) \\
&\leqslant \left\{\frac{p!}{[r(U)-r(\mu^e)]^p}\right\}\frac{\|T\|^2}{r(U)-|r(\mu_e)|-\|T\|} \\
&\quad+\beta^{-1}\exp(-\beta[r(U)-|r(\mu^e)|) \qquad \text{(A.14)}
\end{aligned}
$$

Proof of Estimate b. We start from the estimate a(ii) in which we sum over the graphs of the basis including the winding loops without jump. Next we performed the following summations.

- We sum the hoppings involved in the constituent parts of the linked winding loops, keeping their times' jumps fixed. We use the Lemma A.2.1, in which we replace every hopping appearing in the estimate a(ii) by $\|T\|$.

- We sum over the s.d. of the bubble parts and of the linked winding loops, provided that their numbers and their relative positions are fixed. The S.I. condition implies in particular that $r(U)-\|T\|>0$.

- We sum over the standard jumps, the bubble parts and the number of non winding loops by using the estimate (iii). We notice that $n \geqslant |W|$.

$$
\begin{aligned}
|\Phi[T,U,\mu_e]| &\leqslant \beta^{-1}\sum_{|W|=1}^{|W|=\infty}\exp-\beta\,|W|[r(U)-|r(\mu_e)|] \\
&\quad\times\left\{\sum_{n=|W|+1}^{n=\infty}\frac{\left[\beta\,|W|\times\frac{r(U)\times\|T\|}{r(U)-\|T\|}\right]^n}{n!}\right\} \\
&\leqslant \beta^{-1}\sum_{|W|=1}^{|W|=\infty}\exp-\beta\,|W|\left\{r(U)-|r(\mu_e)|-\frac{r(U)\times\|T\|}{r(U)-\|T\|}\right\} \\
&\leqslant C\beta^{-1}\exp-\beta\left\{r(U)-|r(\mu_e)|-\frac{r(U)\times\|T\|}{r(U)-\|T\|}\right\} \qquad \text{(A.15)}
\end{aligned}
$$

- Next we suppose that $\beta\,\frac{\|T\|^{p+1}}{[r(U)-r(\mu^e)]^p} > A_p$. Then there are two positive constants C_p and C'_p:

$$
|\Phi[U,T,\mu_e]| \leqslant C'_p\beta^{-1} \leqslant C_p\,\frac{\|T\|^{p+1}}{[r(U)-\|T\|]^p}
$$

This concludes the proof.

APPENDIX B. THE DOBRUSHIN'S INEQUALITIES

The study of convergent cluster expansion on a lattice in the classical statistical mechanics is done in many papers see for example refs. 21–24. Our goal is to prove the existence of a convergent C.E. for every conditional partition function $Z(H; \Sigma_V \mid \Sigma_{\bar{V}})$. We extend one of the last papers of Dobrushin,[21] partly devoted to the C.E. in the classical case, to the quantum case. Dobrushin's approach relies on the existence of an upper bound for the logarithm of the ratio of two partition functions associated to two different volumes, here a volume means a set of loops. The main difficulty in the quantum case stems from the continuous time appearing in the F.K. representation. We use the lattice approximation introduced in the Appendix A. To prove the "discrete" Dobrushin's inequality, for finite S, we choose a real positive bounded function $B_S(\hat{\lambda}_S)$, which depends of the length and of the number of jumps of the loop.

$$B_S(\hat{\lambda}) = \exp\left\{ a(H) \times \frac{|\hat{\lambda}_S|}{S} + b(H)\, |J(\hat{\lambda}_S)| \right\}$$

$|\hat{\lambda}^S|$ is the sum of the lengths of the vertical segments of the loop $\hat{\lambda}^S$, $|J(\hat{\lambda})|$ is the number of jumps of $\hat{\lambda}_S$. $a(H)$ and $b(H)$ are bounded positive functions. Notice that $\lim_{S \to \infty} \frac{|\hat{\lambda}_S|}{S} = |\hat{\lambda}|$. The following limit exists:

$$B(\hat{\lambda}) = \lim_{S \to \infty} B_S\,(\hat{\lambda}_S) = \exp\{a(H) \times |\hat{\lambda}| + b(H)\, |J(\hat{\lambda})|\}$$

Proposition B.1. The "discrete" Dobrushin's inequality. The hamiltonian $\mathbf{H} \in \mathbf{F}$; $\mathbf{J}_{X,Y} = 0$ satisfies an S.I. condition (0.11).

$$|r(\tilde{\mu}^e)| < r(\tilde{U}) - \|T\| - \exp - \beta[r(\tilde{U} - |r(\tilde{\mu}^e)| - \|T\|] + h.o.$$

(i) There exist two functions $a(H)$, $b(H)$, and two strictly positive constants C_1, C_2:

$$a(H) =: \|T\| \left\{ \frac{\|T\|}{r(U)} + \beta\, \|T\| \exp - \beta[r(U) - |r(\mu_e)| - \|T\|) + h.o.] \right\} < C_1\, \|T\| \tag{B.1}$$

$$b(H) =: \left[\frac{\|T\|}{r(U) - r(\mu^e)} \right]^2 + \exp - \beta \left[r(U) - |r(\mu_e)| - \frac{\|T\|}{U} \right] < C_2$$

s.t., for every arbitrary "volume" $\mathscr{V}_S \subset \mathscr{COND}_S(H, \Sigma_V \mid \Sigma_{\bar{V}})$, the discrete Dobrushin's inequality holds:

$$\left|\ln\left\{\frac{Z_{\mathscr{V}_S}(H;\Sigma_V \mid \Sigma_{\bar{V}})}{Z_{\mathscr{V}_S/\hat{\lambda}^i_S}(H;\Sigma_V \mid \Sigma_{\bar{V}})}\right\}\right| \leqslant \exp\left[C_1 \|T\| \frac{|\hat{\lambda}^i_S|}{S} + C_2 |J(\hat{\lambda}^i_S)|\right] \times |\varphi(\hat{\lambda}^i_S)| \qquad \text{(B.2)}$$

(ii) The partition function $Z_{\mathscr{V}_S}$ is an analytic function of the d.s.p. $\varphi(\hat{\lambda}^i_S)$ and $\varphi(\hat{\lambda}^{\{W,i\}}_S)$. Then $Z_{\mathscr{V}_S}(H;\Sigma_V \mid \Sigma_{\bar{V}})$ can be expanded into a convergent C.E. in the d.s.p. of the loops.

Proof. Part (i). To prove the inequality, we have to prove the condition contained in the hypothesis of the first theorem of ref. 21. It is easier to prove the stronger K.P. condition (22), We have to find the conditions under which, the following uniform K.P. condition (w.r.t. the conditional ensemble) is true:

$$\sum_{\{\hat{\lambda}^o_S \in \mathscr{COND}(\Sigma_V \mid \Sigma_{\bar{V}}) \mid \hat{\lambda}^o_S \nsim \hat{\lambda}^1_S\}} |\varphi(\hat{\lambda}^o_S)| \exp\left[a(H) \frac{|\hat{\lambda}^o_S|}{S} + b(H) |J(\hat{\lambda}^o_S)|\right]$$

$$\leqslant a(H) \times \frac{|\hat{\lambda}^1_S|}{S} + b(H) |J(\hat{\lambda}^1_S)| \qquad \text{(B.3)}$$

The symbol $\nsim$ means that the loops $\hat{\lambda}^1_S$ and $\hat{\lambda}^o_S$ are incompatible, i.e., that they intersect along at least one vertical segment. This means that there exists one vertical segment of length $\Delta_S(l_1) =: (l_1 - l'_1)$ of $\hat{\lambda}^1_S$ and one vertical segment of length $\Delta_S(l_o) =: (l_o - l'_o)$ of $\hat{\lambda}^o_S$, which intersect. Let us fix at first $\Delta_S(l_1)$, then we shift the loop $\hat{\lambda}^o_S$ in the time's direction keeping the intersection property between $\hat{\lambda}^1_S$ and $\hat{\lambda}^o_S$, then the corresponding range of variation of l_o is $\Delta_S(l_1) + \Delta_S(l_o)$. In other words, when two loops intersect along two segments $\Delta_S(l_o)$ and $\Delta_S(l_1)$, we have to insert a factor $[\Delta_S(l_o) + \Delta_S(l_1)]$ in the integrand. This fact will be used repetitively in the resummation process. The sum of the s.p. of the loops incompatible with $\hat{\lambda}^1_S$ splits into two parts: the first one, which does not contain the loops of $\mathscr{L}^w_o$, is proportional to $|\hat{\lambda}^1_S|$; the second one, which contains the loops of $\mathscr{L}^w_o$, is proportional to $|J(\hat{\lambda}^1_S)|$.

$$\sum_{\{\hat{\lambda}^o_S \in \mathscr{COND}(\Sigma_V \mid \Sigma_{\bar{V}}) \mid \hat{\lambda}^o_S \nsim \hat{\lambda}^1_S\}} |\varphi(\hat{\lambda}^o_S)| \exp\left[a(H) \frac{|\hat{\lambda}^o_S|}{S} + b(H) |J(\hat{\lambda}^o_S)|\right]$$

$$\leqslant \frac{|\hat{\lambda}^1_S|}{S} \left\{ \sum_{\{\hat{\lambda}^o_S \mid b_o \in \hat{\lambda}^o_S\}}^{*} S \times \exp\left[a(H) \frac{|\hat{\lambda}^o_S|}{S} + b(H) |J(\hat{\lambda}^o_S)|\right] |\varphi(\hat{\lambda}^o_S)| \right\}$$

$$+|J(\hat{\lambda}_S^1)|\left\{\left[\sum_{\{\hat{\lambda}_S^o \mid b_o \in \hat{\lambda}_S^o\}}^{*}|\varphi(\hat{\lambda}_S^o)|\times \Delta_S(l_1)\exp\left[a(H)\frac{|\hat{\lambda}_S^o|}{S}+b(H)\,|J(\hat{\lambda}_S^o)|\right]\right]\right.$$

$$\left.+\exp-\beta(r(U)-|r(\mu^e)|)\right\}$$

$$\leqslant|\hat{\lambda}_1|\,\{\Phi^o[\exp\{b(H)\}\times\|T\|,\, r(U)-a(H)]$$

$$+\Phi^o[\exp\{b(H)\}\times|T|,\, r(U)-a(H),\, r(\mu_e)]\}$$

$$+|J(\hat{\lambda}^1)|\,\{\Phi^1[\exp\{b(H)\}\times\|T\|,\, r(U)-a(H)]$$

$$+\Phi^1[\exp\{b(H)\}\times\|T\|,\, r(U)-a(H),\, r(\mu_e)]\} \tag{B.4}$$

$\sum^*$ means that the sum does not contain the loops of $\mathscr{L}_o^1$. In the two last lines we used the fact that the Riemann sums are bounded by the corresponding integral. To get the fourth line of (B.3), we have used the estimates contained in a(i) and in a(ii) of the Proposition A.1, in which we have made the following changes: $U \to U-a(H)$, and $\|T\| \to \exp\{b(H)\}\times\|T\|$. So that we suppose that the coupling constants of the hamiltonian **H** satisfy a stronger condition S.I., which is defined by doing the corresponding changes. The K.P. condition, that we have to satisfy, is now equivalent to find the two functions $a(H)$ and $b(H)$, which fulfill two inequalities:

$$\begin{aligned} X[a(H),b(H)] =:{}& \Phi^o[\exp\{b(H)\}\|T\|,\, r(U)-a(H)] \\ &+\Phi^o[\exp\{b(H)\}\|T\|,\, r(U)-a(H),\, r(\mu_e)] \leqslant a(H) \\ Y[a(H),b(H)] =:{}& \frac{\Phi^1[\exp\{b(H)\}\|T\|,\, [r(U)-a(H)]]}{r(U)-\exp\{b(H)\}\|T\|-a(H)} \\ &+\Phi^1[\exp\{b(H)\}\|T\|,\, r(U)-a(H),\, r(\mu_e)] \leqslant b(H) \end{aligned} \tag{B.5}$$

For all the finite fixed values of the coupling constants satisfying the S.I. condition. $X[a(H), b(H)]$ and $Y(a(H), b(H)]$ are strictly positive increasing functions of $a(H)$ and $b(H)$, both start from a positive value at $a(H)=0$ and goes to infinity for $a(H)=r(U)-\exp\{\overline{b(H)}\}\times\|T\|$. We want to find the smallest value of $a(H)$: $\overline{a(H)}$ and of $b(H)$: $\overline{b(H)}$ provided that an S.I. condition holds. From simple algebraic computations, we find that there exists two positive constants $C_1<1$ and $C_2<1$, such that the K.P. condition is satisfied:

$$\overline{a(H)} = C_1\times\|T\|, \qquad \overline{b(H)} = C_2 \tag{B.6}$$

The final S.I. condition is defined from the previous S.I. condition by making the following changes:

$$U \to U - C_1 \|T\|; \qquad \|T\| \to \|T\| \exp(C_2)$$

This concludes the proof of the validity of the discrete Dobrushin's inequality.

(ii) To get the last statement of the proposition, we iterate the discrete Dobrushin's inequality to the set of the loops of Λ_S. Then, for every fixed S, every discrete conditional partition function is an analytic function of the family of the s.p. $\varphi(\hat{\lambda}_S^i)$ and $\varphi(\hat{\lambda}_S^{\{W,i\}})$ of the loops contained in Λ_S.[21]

Here comes the major difference between the classical C.E. and the quantum C.E.: the discrete Dobrushin's inequality obtained by removing one loop $\hat{\lambda}_S^i$ from $\mathscr{V}_S$, is not enough to prove the analyticity of the limit $S \to \infty$ of the conditional partition function, because of course the s.p. of a discrete loop goes to zero when $S \to \infty$. Nevertheless the probability densities could be defined in the limit $S \to \infty$ if an S.I. condition holds. Another way to get a meaningful Dobrushin's inequality is to apply the discrete Dobrushin's inequality to a complete volume $\mathscr{V} \subset \mathscr{CONP}(H; \Sigma_V \mid \Sigma_{\bar{V}})$ built from the unions over the unit times intervals of two sets of complete families of loops: the first one $\{\mathscr{F}_{\{o,1\}}[\vec{A}_1^P] \cdots \mathscr{F}_{\{o,1\}}(\vec{A}_p^P)\}$ is composed of non winding loops, and the second one of winding loops: $\{\mathscr{F}^w_{\{o,1\}}(\vec{A}_1^P), \ldots, \mathscr{F}_{\{o,1\}}(\vec{A}_q^P)\}$. It is convenient to distinguish by a site index the rescaled hopping $\frac{t_{A_i}}{U}$, and the chemical potential μ_j^e associated to the coupling constants corresponding to these complete families. ▌

Proposition B.2. The "quantum" Dobrushin's inequality. The hypothesis of the Proposition B.1.1. are satisfied.

(i) Then, for every complete volume $\mathscr{V} \subset \mathscr{CONP}(H; \Sigma_V \mid \Sigma_{\bar{V}})$ built from the complete families $\{\mathscr{F}_{[o,\beta)}(\vec{A}_1^P), \ldots, \mathscr{F}_{[o,\beta)}(\vec{A}_p^P); \mathscr{F}^w_{[o,\beta)}(\vec{A}_1^P), \ldots, \mathscr{F}^w_{[o,\beta)}(\vec{A}_q^P)\}$, the "quantum" Dobrushin's inequality holds:

$$\left| \mathrm{Ln} \left\{ \frac{Z_{\mathscr{V}}(H; \Sigma_V \mid \Sigma_{\bar{V}})}{Z_{\mathscr{V}/\mathscr{F}_{[o,1]}(\vec{\mathscr{A}}_i^{\mathscr{P}})}(H; \Sigma_V \mid \Sigma_{\bar{V}})} \right\} \right| \leqslant \frac{\prod_{j=o}^{j=n} \exp(C_2) \times |t_{A^j}|}{|r(U) - C_1 \|T\||^n} \tag{B.7}$$

(ii) $Z_{\mathscr{V}}(H; \Sigma_V \mid \Sigma_{\bar{V}})$ is an analytic function of the two sets of variables $\{\frac{t_{A_i}}{U}\}_i$, $\{\exp -\beta[U - \mu_j^e]\}_j$.

Proof. Part (i). We apply the discrete Dobrushin's inequality to the quotient of the two partition functions defined in the two "volumes" $\mathscr{V}_{\mathscr{S}}$ and $\mathscr{V}_{\mathscr{S}}/\mathscr{F}^{S}_{[0,1]}(\vec{A}^{P}_{i})$:

$$\left|\mathrm{Ln}\left\{\frac{Z_{\mathscr{V}_{\mathscr{S}}}(H;\Sigma_V\mid\Sigma_{\bar{V}})}{Z_{\mathscr{V}_S/\mathscr{F}^{S}_{[o,1]}(\vec{A}^{P}_{i})}(H;\Sigma_V\mid\Sigma_{\bar{V}})}\right\}\right|$$
$$\leqslant\left[\prod_{j=o}^{j=n}\exp(C_2)\times|t_{A^j}|\right]\sum_{b_o=o}^{b_o=S}\frac{1}{S}\sum_{b_n=n}^{b_n=\infty}\frac{1}{S}e^{-[(r(U)-C_1\|T\|]\frac{b_n}{S}}\sum_{b_{n-1}=n-1}^{b_{n-1}=b_n}\frac{1}{S}\cdots\sum_{b_1=1}^{b_1=b_2}\frac{1}{S} \tag{B.8}$$

The R.H.S is a multiple Riemann sum bounded from above for every S, by the corresponding integral provided that the S.I. condition is satisfied. Then we take the limit $S\to\infty$ of the L.H.S. to get the "quantum" Dobrushin's inequality.

$$\lim_{S\to\infty}\left|\mathrm{Ln}\left\{\frac{Z_{\mathscr{V}^S}(H;\Sigma_V\mid\Sigma_{\bar{V}})}{Z_{\{\mathscr{V}_S/\mathscr{F}^{S}_{[o,1]}(\vec{A}^{P}_{i})\}}(H;\Sigma_V\mid\Sigma_{\bar{V}})}\right\}\right|$$
$$\leqslant\prod_{i=o}^{i=n}e^{C_2}\,|t_{A^i}|\left\{\int_o^1 dt_o\int_0^\infty e^{-t_1[r(U)-C_1\|T\|]}\,dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\right\} \tag{B.9}$$

$$\left|\mathrm{Ln}\left\{\frac{Z_{\mathscr{V}}(H;\Sigma_V\mid\Sigma_{\bar{V}})}{Z_{\{\mathscr{V}/\mathscr{F}_{[o,1]}(\vec{A}^{P}_{i})\}}(H;\Sigma_V\mid\Sigma_{\bar{V}})}\right\}\right|\leqslant\frac{\prod_{i=o}^{i=n}\exp(C_2)\times|t_{A^i}|}{|r(U)-C_1\,\|T\||^n}$$

Part (ii). By iterating the inequality (B.8) over all the complete families, which compose $\mathscr{V}$, we deduce that every conditional partition function $Z(H;\Sigma_V\mid\Sigma_{\bar{V}})$ is uniformly bounded provided that the above S.I. condition is satisfied. Then every conditional partition function is an analytic function in the weights of the complete families, i.e., in the two families of variables $\{[\frac{t_{A_i}}{U}]\}_i$; $\{\exp-\beta[U-\mu^e_j]\}_j\}$ for every fixed β. ∎

APPENDIX C. THE CLUSTER EXPANSION IN THE SPACE

In the previous section, we have proved the existence of a convergent C.E. for every discrete conditional partition function $Z_S(H;\Sigma_V\mid\Sigma_{\bar{V}})$. Next we want to prove the existence, for every hamiltonian **H**, of an effective hamiltonian $\mathscr{H}_\beta$ defined on the basis Z^ν, which potentials decay exponentially. This will be done by performing a time and a space resummation of the truncated functions. The reference to the conditional ensemble and to the hamiltonian **H** appears only in the local structure of the cluster itself.

3.1. The Cluster Expansion in the Space Time

3.1.1. Definitions: Clusters (Fig. 3)

- A cluster $C^S[T, \vec{\mathscr{A}}^{\mathscr{P}}]$ is built from a maximally connected family of intersecting loops $\{\hat{\lambda}_S^1,\ldots, \hat{\lambda}_S^p\}$, and from a set of integers $\{\alpha(\hat{\lambda}_S^1)\cdots\alpha(\hat{\lambda}_S^p)\}$, which are the multiplicities of the loops in the cluster, $\{T\}$ is the family of jump's times, and $\mathscr{A}^{\mathscr{P}} \sim \{A_o^{P_o},\ldots, A_n^{P_n}\}$ is the set of the supports of the correspondent hoppings with a given pairings $\mathscr{P}$. We denote by $C^{\{S, w\}}[T, \vec{\mathscr{A}}^{\mathscr{P}}]$ a cluster, which contains at least one winding loop.

3.1.2. Definitions: Projections of the Clusters on V

The projection of the cluster $C^S\{T, \vec{\mathscr{A}}^{\mathscr{P}}\}$ on V is a set of l.i.d. graphs $\vec{\mathscr{A}}^{\mathscr{P}} \sim \{\vec{A}_1^P,\ldots, \vec{A}_n^P\}$, each $\vec{A}_i^P$ is the projection of a constituent linked loops building the cluster.

- Two clusters $C^S[T, \mathscr{A}^{\mathscr{P}}] \subset \mathscr{V}_S$ and $C^S(T', \mathscr{A}^{\mathscr{P}}] \subset \mathscr{V}_S$ are equivalent ($C^S[T, \mathscr{A}^{\mathscr{P}}] \sim C^S[T', \mathscr{A}^{\mathscr{P}}]$), if their projections on the basis are the same linked graph.
- $\vec{\mathscr{A}}^{\mathscr{P}} \to \mathscr{A}^{\mathscr{P}}$ is the set of the directed graphs, which support are the same unoriented graph.
- The complete family of clusters $\mathscr{F}^S_{[a, a+1)}(\vec{\mathscr{A}}^{\mathscr{P}})$ (resp. $\mathscr{F}^S_{[a, a+1)}(\mathscr{A}^{\mathscr{P}})$) is the set of the equivalent clusters projected on the same graph $\vec{\mathscr{A}}^{\mathscr{P}}$ (resp. the same unoriented graph $\mathscr{A}^{\mathscr{P}}$), which birth's times belong to $[0, 1)$.

Now we suppose that an S.I. condition holds, such that the discrete Dobrushin's inequality holds. Then every discrete conditional partition function $Z_{\mathscr{V}_S}$ is an analytic function of the d.s.p. $\varphi(\hat{\lambda}_S^i)$ of the loops contained in $\mathscr{V}_S$, provided that S is finite. Next we can expand, via the Taylor formula, the logarithm of the partition function[21] in term the d.s.p. of the loops, leading to the definition of the truncated functions:

$$\begin{aligned}
\ln Z_{\mathscr{V}_S}[((H; \Sigma_V \mid \Sigma_{\bar{V}})] &= F_{\mathscr{V}_S}\{\hat{\lambda}_S^1,\ldots, \hat{\lambda}_S^p\} \\
&:= \sum_{C^S[T, \mathscr{A}^{\mathscr{P}}] \in \mathscr{V}_S} R(\vec{\mathscr{A}}^{\mathscr{P}}) \prod_{i=1}^{i=p} [\varphi(\hat{\lambda}_S^i)]^{\alpha(\hat{\lambda}_S^i)} \\
&:= \sum_{C^S[T, \mathscr{A}^{\mathscr{P}}] \in \mathscr{V}_S} \phi(C^S[T, \mathscr{A}^{\mathscr{P}}]) \\
&= \sum_{\{\vec{\mathscr{A}}^{\mathscr{P}} \to \mathscr{A}^{\mathscr{P}}\}} \left\{ \sum_{\{C^S[T, \mathscr{A}^{\mathscr{P}}] \to \vec{\mathscr{A}}^P\}} \phi(C^S_{\{T, \mathscr{A}^{\mathscr{P}}\}}) \right\} \qquad \text{(C.1)}
\end{aligned}$$

The coefficients $R(\vec{\mathscr{A}}^{\mathscr{P}})$ depend of the geometry of the cluster $C^S[T, \vec{\mathscr{A}}^{\mathscr{P}}]$ only, and then of $\vec{\mathscr{A}}^{\mathscr{P}}$, they will be bounded later. We will omit the subscript "S" The truncated functions of the cluster $C[T, \mathscr{A}^{\mathscr{P}}]$, are built from the partition function restricted to the constituent loops of the cluster $C[T, \mathscr{A}^{\mathscr{P}}]$.[21]

Notes.

(A) The truncated function of the cluster $C[T, \mathscr{A}^{\mathscr{P}}]$, is built from the partition function restricted to the constituent loops of the cluster $C[T, \mathscr{A}^{\mathscr{P}}]$ ONLY ref. 21.

(B) One of the main differences between the C.E. in continuous cases and the C.E. in the discrete case is that only the truncated functions defined on the continuous clusters composed of loops with multiplicity one contribute. It is straightforward to see that the sum over the jump's times of every power larger than one of a d.s.p. of a loop goes to zero, when $S \to \infty$. Notice the exception of the linked loops of $\mathscr{L}_o^1$.

3.1.3. Definitions: Trees in the Space Time

• Every constituent loop of a cluster $C[T, \vec{\mathscr{A}}^{\mathscr{P}}]$ is identified with a point of the space time. Two points are connected if the corresponding loops intersect. This construction is iterated to all the constituent loops of the cluster. We built the dual graph $C^*[T, \mathscr{A}^{\mathscr{P}}] \subset \Lambda_S$. Let $\mathscr{T}(\vec{\mathscr{A}}^{\mathscr{P}})$ be the set of spanning trees built from $C^*[T, \mathscr{A}^{\mathscr{P}}]$.[23] Infinitely many clusters have the same spanning tree $\tau(\vec{\mathscr{A}}^{\mathscr{P}}) \in \mathscr{T}(\vec{\mathscr{A}}^{\mathscr{P}})$, which come out by varying the jumps' times of its constituent intersecting loops. The clusters of $\mathscr{F}_{[o,1)}(\vec{\mathscr{A}}^{\mathscr{P}})$ are decomposed into the $\tau(\vec{\mathscr{A}}^{\mathscr{P}})$ equivalent classes $\{C[T, \vec{\mathscr{A}}^{\mathscr{P}}] \to \tau(\vec{\mathscr{A}}^{\mathscr{P}})\}$, which are the clusters with the same spanning tree.

3.2. The Space-Time Summation

3.2.1. Definitions: Towards the Effective Potentials

It will be convenient to make a three steps summation of the truncated functions to get the expression of the effective potentials. Notice that the birth's time of the cluster will be integrated over the interval $[o, 1)$.

Step I. The sum over the clusters with the same spanning tree.

$$\phi^{\beta}[\tau(\vec{\mathscr{A}}^{\mathscr{P}})] = \sum_{\{C[T, \vec{\mathscr{A}}^{\mathscr{P}}] \to \tau(\vec{\mathscr{A}}^{\mathscr{P}})\}} \phi(C[T, \vec{\mathscr{A}}^{\mathscr{P}}]) \qquad \text{(C.2)}$$

Step II. The sum over the spanning trees built from $\mathscr{T}(\vec{\mathscr{A}}^{\mathscr{P}})$.

$$\tilde{\phi}^{\beta}[\mathscr{T}(\vec{\mathscr{A}}^{\mathscr{P}})] = \sum_{\{\tau(\vec{\mathscr{A}}^{\mathscr{P}}) \in \mathscr{T}(\vec{\mathscr{A}}^{\mathscr{P}})\}} \phi^{\beta}[\tau(\vec{\mathscr{A}}^{\mathscr{P}})] \tag{C.3}$$

Step III. The sum over the potentials, which have the same family of sets. We have, in particular, to sum over all the pairings of $\mathscr{A}$, (we add the subscript "w" when the cluster contains a winding loop).

$$\Psi^{\beta}[\Sigma_{\mathscr{A}}] = \sum_{\{\vec{\mathscr{A}}^{\mathscr{P}} \to \mathscr{A}\}} \phi^{\beta}[\mathscr{T}(\vec{\mathscr{A}}^{\mathscr{P}})] \tag{C.4}$$

Definition. The sum of the potentials, which clusters contain at least one winding loops is written:

$$R_p^{\{\beta, w\}}[\Sigma_V \mid \Sigma_{\bar{V}}] = \sum_{\{\Sigma_{\mathscr{A}} \in \Sigma_V \cup \Sigma_{\bar{V}} \mid o \in \mathscr{A}, |\mathscr{A}| \leqslant p\}} |\Psi_{\mathscr{A}}^{\beta, w}[\Sigma_V \mid \Sigma_{\bar{V}}]| \tag{C.5}$$

Proposition C.2.2. The Hamiltonian $\mathbf{H} \in \mathbf{F}$; $\mathbf{J}_{X,Y} = 0$ fulfills an S.I. condition.

(a) Then the following general bounds hold:

$$|\Psi_p^{\beta}[\Sigma_{\mathscr{A}}]| \leqslant [r(\tilde{U}) - |(\tilde{\mu}_e)|] \times \prod_{i=2}^{i=n} \left[\frac{|A_i|}{2}\right]! \times \frac{|t_{A^j}|}{r(\tilde{U}) - |r(\tilde{\mu}_e)| - \|T\|} \tag{C.6}$$

$$\Psi^{\beta}[\Sigma_v] = \beta^{-1} \log[1 + \exp(-\beta(U - \sigma_v\, |\mu^e|)]$$

(b) Moreother for $\beta \frac{\|T\|^{p+1}}{[r(U) - r(\mu^e)]^p} > A_p$ and for $|\mathscr{A}| < p$, we get:

$$|R_p^{\{\beta, w\}}[\Sigma_V \mid \Sigma_{\bar{V}}]| \leqslant \frac{\|T\|^{p+1}}{[r(\tilde{U}) - |r(\tilde{\mu}_e)| - \|T\|]^p} \tag{C.7}$$

Before going to the proof of the proposition, we prove the corollary:

Corollary C.2.3. The hamiltonian $\mathbf{H} \in \mathbf{F}$; $\mathbf{J}_{X,Y} = 0$ fulfills an S.I. condition, then we get:

$$\sum_{\{\Sigma_{\mathscr{A}} \in \Sigma_V \cup \Sigma_{\bar{V}} \mid o \in \mathscr{A}, |\mathscr{A}| \leqslant p\}} |\Psi^{\beta}[\Sigma_{\mathscr{A}}]| \leqslant \infty \tag{C.8}$$

Proof. We start from the estimate (A) of the Proposition C.1.2.

- We sum over all the hoppings. We use the Lemma A.2.1), we replace $[\frac{|A_i|}{2}]! \times |t_{A_i}|$ by $\|T\|$.
- We sum over the number of hoppings.

$$\sum_{\{\Sigma_{\mathscr{A}} \in \Sigma_V \cup \Sigma_{\bar{V}} \mid o \in \mathscr{A}, |\mathscr{A}| \leq p\}} |\Psi^{\beta}[\Sigma_{\mathscr{A}}]|$$

$$\leq \|T\| \times \sum_{n=1}^{n=\infty} \left[\frac{\|T\|}{r(\tilde{U}) - |r(\tilde{\mu}_e)|} \right]^n + \beta^{-1} \log[1 + \exp(-\beta(U - \sigma_v |\mu^e|)] \tag{C.9}$$

$$\leq \frac{\|T\|^2}{r(\tilde{U}) - |r(\tilde{\mu}_e)| - \|T\|} + \beta^{-1} \log[1 + \exp(-\beta(U - \sigma_v |\mu^e|)]$$

The chemical potential is infinite for $\beta = 0$, this is harmless because we can withdraw, for small β, from the chemical potential the term β^{-1} per vertex, then the sum will be finite.

Proof of the Proposition C.2.2. We adapt the usual method of resummation of clusters, done in the classical case (see, for example, ref. 23, p. 976), to the quantum case with the following changes.

- First two intersecting loops of a cluster intersect along vertical segments.
- Secondly the sum over the weights of the contours containing one point, has to be replaced by an integral over the jump's times of the s.p. of the loops. The sum is done in three steps:

Step I. the sum of the s.d. of the loops belonging to the complete clusters with the same tree τ.

Step II. the sum over the spanning trees of the dual graph $C^*[T, \vec{\mathscr{A}}^{\mathscr{P}}]$.

Step III. the sum over the different loop's decompositions of the families of clusters, which have the same set of jumps, together with the sum over the pairings.

3.2.2. Step I. The Summation over the S.P. of the Clusters $C[T, \vec{\mathscr{A}}^{\mathscr{P}}] \to \tau(\vec{\mathscr{A}}^{\mathscr{P}})$

The sum of the truncated functions of the clusters of $C[T, \vec{\mathscr{A}}^{\mathscr{P}}] \to \tau(\vec{\mathscr{A}}^{\mathscr{P}})$ is done according to a prescribed order. The time's integrations are always performed over the active times.

• We first integrate the s.p. of the loops of the zeroth generation $\{\hat{\lambda}_1^o \cdots \hat{\lambda}_{a_o}^o\}$, by using the estimates a(1) and a(2) of the Proposition A.2 keeping the intersection property. These families of loops are in one to one correspondence with the vertices of the tree $\tau(\vec{\mathscr{A}}^{\mathscr{P}})$, which have incidence number one.

• Let $\tau_1(\vec{\mathscr{A}}^{\mathscr{P}}) \subset \tau(\vec{\mathscr{A}}^{\mathscr{P}})$ be the tree obtained by removing from $\tau(\vec{\mathscr{A}}^{\mathscr{P}})$ its boundary edges. Then we proceed by integrating the s.p. of the loops of the first generation $\{\hat{\lambda}_1^1 \cdots \hat{\lambda}_{a_1}^1\}$, which are in one to one correspondence with the vertices with incidence one in $\tau_1(\vec{\mathscr{A}}^{\mathscr{P}})$.

• $\tau_n(\vec{\mathscr{A}}^{\mathscr{P}}) \subset \tau_{n-1}(\vec{\mathscr{A}}^{\mathscr{P}})$ is the last iteration of the process, the tree is now reduced to one point, which is the loop $\hat{\lambda}_1^n$, we integrate the s.p. of $\hat{\lambda}_1^n$ including its birth's time, which belongs to the interval $[o, 1)$.

We follow the previous scheme, we begin by the integration of the s.d. of the loops of the zeroth generation $\Lambda^o =: \{[\hat{\lambda}_{\{1,1\}}^o \cdots \hat{\lambda}_{\{1,\alpha_1\}}^o],\ldots, [\hat{\lambda}_{\{2,1\}}^o \cdots \hat{\lambda}_{\{2,\alpha_2\}}^o] \cdots [\hat{\lambda}_{\{r,1\}}^o \cdots \hat{\lambda}_{\{r,\alpha_r\}}^o]\}$, which intersect a given loop $\hat{\lambda}_1^1$ of the first generation along one of its segment say $\Delta(t^1)$. This family is partitioned into r subfamilies, each subfamily is composed of loops belonging to the same complete family. Let $\Delta(t^1)$ be the intersection between the loop $\hat{\lambda}_1^1$ and between the loops of Λ^o. The corresponding segments of the loops of Λ^o are denoted by:

$$T^o =: \{[\Delta(t_{\{1,1\}}^o) \cdots \Delta(t_{\{1,\alpha_1\}}^o)],\ldots, [\Delta(t_{\{2,1\}}^o) \cdots \Delta(t_{\{2,\alpha_2\}}^o)] \cdots [\Delta(t_{\{r,1\}}^o) \cdots \Delta(t_{\{r,\alpha_r\}}^o)]\}$$

We need to estimate the following quantity:

$$S[\Lambda^o \nprec \lambda^1] =: \frac{1}{\alpha^1! \times \cdots \times \alpha^r!} \varphi(\lambda^1)$$

$$\to \prod_{i=1}^{i=\alpha_1} \times \int \varphi(\lambda_{\{1,i\}}^o)\, d(\lambda_{\{1,i\}}^o) \times \cdots \times \prod_{i=1}^{i=\alpha_r} \int \varphi(\lambda_{\{r,i\}}^o)\, d(\lambda_{\{r,i\}}^o) \quad \text{(C.10)}$$

The symbol $\to$ in the integral means that the loop λ^1 intersects each loop of Λ^o in the way described above. The factorials, which are in front of the R.H.S., stem from the fact that, after the integration of the s.p., the loops belonging to the same complete family are indistinguishable. Now we use the fact contained in the proof of the Proposition B.1: the intersection property between the loops λ^1 and $\lambda_{\{1,i\}}^o$ is equivalent to insert a factor $[\Delta(t^1 + \Delta(t_{\{1,i\}}^o)]$ in the integrand:

$$S[\Lambda^o \nprec \lambda^1] = \varphi(\lambda^1) \times \frac{1}{\alpha^1! \times \cdots \times \alpha^r!}$$

$$\times \Bigg[\prod_{i=1}^{i=\alpha_1} \int \varphi(\lambda^o_{\{1,i\}}) [\Delta(t^1) + \Delta(t^o_{\{1,i\}})] \, d(\lambda^o_{\{1,i\}})$$

$$\times \cdots \times \prod_{i=1}^{i=\alpha_r} \int \varphi(\lambda^o_{\{r,i\}}) [\Delta(t^1) + \Delta(t^o_{\{r,i\}})] \, d(\lambda^o_{\{r,i\}}) \Bigg]$$

$$= \frac{1}{\alpha^1! \times \cdots \times \alpha^r!} \times \sum_{j=o}^{j=\alpha^o} \varphi(\lambda^1) \, \Delta(t^1)^j$$

$$\times \Bigg\{ \sum_{I_j \subset \mathscr{I}_j} \prod_{i=1}^{i=\alpha_1} \int \varphi(\lambda^o_{\{1,i\}}) [\hat{\Delta}(t^o_{\{1,i\}}] \, d(\lambda^o_{\{1,i\}})$$

$$\times \cdots \times \prod_{i=1}^{i=\alpha_r} \int \varphi(\lambda^o_{\{r,i\}}) [\hat{\Delta}(t^o_{\{r,i\}})] \, d(\lambda^o_{\{r,i\}}) \Bigg\} \qquad \text{(C.11)}$$

The set $\mathscr{I}_j$ is the set of the subsets of the set of indices $\{(1,1),\ldots,(1,\alpha_1);\ldots;(r,1),\ldots,(r,\alpha_r)\}$ containing j elements. The summation over $\mathscr{I}_j \subset \mathscr{I}_j$ has the following meaning:

$$\hat{\Delta}(t^o_{\{1,i\}}) = \Delta(t^o_{\{1,i\}}) \quad \text{if } \{1,i\} \in I_j; \qquad \hat{\Delta}(t^o_{\{1,i\}}) = 1 \quad \text{if } \{1,i\} \notin I_j$$

Now we can integrate and bound the s.p. of the loops of Λ^o by using the upper bounds contained in the estimates of the Proposition A.2. It is convenient to use the rescaled hoppings: $T_{A^j} := \frac{|t_{A^j}|}{r(\tilde{U}) - |r(\tilde{\mu}_e)|}$.

$$S[\Lambda^o \nprec \lambda^1] \leqslant \sum_{j=o}^{j=\alpha_1+\cdots+\alpha_r} \varphi(\lambda^1) \, \Delta(t^1)^j \times (r(\tilde{U}) - |r(\tilde{\mu}_e)|)^j$$

$$\times \frac{C^j_{(\alpha_1+\cdots+\alpha_r)}}{\alpha^1! \times \cdots \times \alpha^r!} \left\{ \prod_{i=o}^{i=m^1} |T_{A^o_{\{1,i\}}}| \right\}^{\alpha^1} \times \cdots \times \left\{ \prod_{i=0}^{i=m^r} |T_{A^o_{\{r,i\}}}| \right\}^{\alpha^r}$$

$$\leqslant \Bigg[\sum_{j=o}^{j=\alpha_1+\cdots+\alpha_r} \varphi(\lambda^1) \, \Delta(t^1)^j \times \frac{(r(\tilde{U}) - |r(\tilde{\mu}_e)|)^j}{j! \times (\alpha_1 + \cdots + \alpha_r - j)!} \Bigg]$$

$$\times \frac{(\alpha_1 + \cdots + \alpha_r)!}{\alpha^1! \times \cdots \times \alpha^r!} \left\{ \prod_{i=o}^{i=m^1} |T_{A^o_{\{1,i\}}}| \right\}^{\alpha^1} \times \cdots \times \left\{ \prod_{i=o}^{i=m^r} |T_{A^o_{\{r,i\}}}| \right\}^{\alpha^r}$$

The next step is to absorb the factorials in the rescaled hoppings, by using the combinatorial lemma.

3.2.5. The Combinatorial Lemma

Let $\{a_1,\ldots, a_n\}$ be a sequence of strictly positive numbers, and let $\{\kappa_1,\ldots, \kappa_n\}$ be a sequence of positive integers. Then the following inequality is true:

$$\frac{\{\kappa_1+\cdots+\kappa_n\}!}{\kappa_1!\times\cdots\times\kappa_n!}<\prod_{i=1}^{i=n}\left[\frac{1}{a_i}\right]^{\kappa_i}\times\left\{\sum_{i=1}^{i=n}a_i\right\}^{(\kappa_1+\cdots+\kappa_n\}} \tag{C.12}$$

Proof. The above inequality is equivalent to:

$$\frac{\{\kappa_1+\cdots+\kappa_n\}!}{\kappa_1!\times\cdots\times\kappa_n!}\times\prod_{i=1}^{i=n}a_i^{\kappa_i}<\left\{\sum_{i=1}^{i=n}a_i\right\}^{(\kappa_1+\cdots+\kappa_n\}}$$

The L.H.S. is one of the terms contained in the expansion of the R.H.S. of the inequality. To compute the coefficient of the product $\prod_{i=1}^{i=n}a_i^{\kappa_i}$ in the R.H.S., we first select the terms containing $a_1^{\kappa_1}$, by choosing κ_1 factors in the product of the R.H.S., there are $C^{\kappa_1}_{\{\kappa_1+\cdots+\kappa_n\}}$ possible choices. Next, to select the term containing $a_2^{\kappa_2}$, we choose κ_2 factors in the $\{\kappa_2+\cdots+\kappa_n\}$ remaining factors of the R.H.S., there are $C^{\kappa_2}_{\{\kappa_2+\cdots+\kappa_n\}}$ possible choices. We iterate the process to get:

$$C^{\kappa_1}_{\{\kappa_1+\cdots+\kappa_n\}}\times C^{\kappa_2}_{\{\kappa_2+\cdots+\kappa_n\}}\times\cdots\times C^{\kappa_n}_{\{\kappa_n\}}=\frac{\{\kappa_1+\cdots+\kappa_n\}!}{\kappa_1!\times\cdots\times\kappa_n!}\quad\blacksquare$$

As we will use repetitively the formula (C.12) of the "combinatorial lemma," we explain in detail how it works. Let $|T_{A^o_{\{k,i\}}}|$ be the maximum of the $|T_{A^o_{\{k,i\}}}|$ when an S.I. condition holds. We make the following choices for a_j and for κ_j:

$$\kappa_j:=\alpha^j;\qquad a_j:=4\cdot|\bar{T}_{A_i^{\{o,j\}}}|$$

The sup is taken over the values of the rescaled hoppings, which satisfy the S.I. condition.

$$\sum_{\{0\in A^i\}}|T_{A^i}|:<\overline{C(H)}<\infty \tag{C.13}$$

This is the sum of the s.p. of the complete family of the loops, which projections on the basis contain one fixed point, it was shown to be uniformly convergent in the Proposition A.1(i) and (ii), if an S.I. condition

(0.11) is satisfied. Then we deduce the following uniform bound w.r.t. every conditional ensemble:

$$\frac{(\alpha^1+\cdots+\alpha^r)!}{\alpha^1!\times\cdots\times\alpha^r!}\leqslant\left[\frac{4\overline{C(H)}}{\prod_{i=o}^{i=m^1}|\bar{T}_{A^o_{\{1,i\}}}|}\right]^{\alpha^1}\times\cdots\times\left[\frac{4\overline{C(H)}}{\prod_{i=o}^{i=m^r}|\bar{T}_{A^o_{\{r,i\}}}|}\right]^{\alpha^r} \tag{C.14}$$

In the expression of $S[\Lambda^o\curlywedge\lambda^1]$, we replace the factorials by the above upper bound to get:

$$\begin{aligned}&\frac{(\alpha_1+\cdots+\alpha_r)!}{\alpha^1!\times\cdots\times\alpha^r!}\left\{\prod_{i=o}^{i=m^1}|T_{A^o_{\{1,i\}}}|\right\}^{\alpha^1}\times\cdots\times\left\{\prod_{i=o}^{i=m^r}|T_{A^1_{\{r,i\}}}|\right\}^{\alpha^r}\\&\qquad\leqslant\left\{\prod_{i=o}^{i=m^1}|\hat{T}^{[1]}_{A^o_{\{1,i\}}}|\right\}^{\alpha^1}\times\cdots\times\left\{\prod_{i=1}^{i=m^r}|\hat{T}^{[1]}_{A^o_{\{r,i\}}}|\right\}^{\alpha^r}\end{aligned} \tag{C.15}$$

Where $|\hat{T}^{[1]}_{A^{[1]}_{\{k,i\}}}|:=4\bar{\Phi}\times|T_{A^o_{\{k,j\}}}|\times|\bar{T}_{A^o_{\{k,j\}}}|^{-1}$

$$\begin{aligned}S[\Lambda^o\curlywedge\lambda^1]&\leqslant\varphi(\lambda^1)(r(\tilde{U})-|r(\tilde{\mu}_e)|)\left[\sum_{j=o}^{j=\alpha_1+\cdots+\alpha_r}\frac{1}{(\alpha_1+\cdots+\alpha_r-j)!}\right]\\&\quad\times\left\{\prod_{i=o}^{i=m^1}|\hat{T}^{[1]}_{A^o_{\{1,i\}}}|\right\}^{\alpha^1}\times\cdots\times\left\{\prod_{i=o}^{i=m^r}|\hat{T}^{[1]}_{A^o_{\{r,i\}}}|\right\}^{\alpha}\\&\leqslant\varphi(\lambda^1)\times(r(\tilde{U})-|r(\tilde{\mu}_e)|)\times e\\&\quad\times\left\{\prod_{i=o}^{i=m^1}|\hat{T}^{[1]}_{A^o_{\{1,i\}}}|\right\}^{\alpha^1}\times\cdots\times\left\{\prod_{i=o}^{i=m^r}|\hat{T}^{[1]}_{A^o_{\{r,i\}}}|\right\}^{\alpha^r}\end{aligned} \tag{C.16}$$

The first inequality of (C.16) was obtained by using the estimates (iii) and (iv) of the Proposition A.2.

Next the time's integration, which leads to the estimate of $S[\Lambda^o\curlywedge\lambda^1]$, can be iterated to the second linked loop of the first generation, and then to all the loops of the second generation. Then the integration over the edges of $\tau_1(\vec{\mathscr{A}}^{\mathscr{P}})\subset\tau(\vec{\mathscr{A}}^{\mathscr{P}})$ are performed according to the order induced by the tree $\tau(\vec{\mathscr{A}}^{\mathscr{P}})$. Next we iterate the integration to all the generations of loops up to the last one, which is reduced to one linked loop. Finally we integrate the jump's times of the last linked loop, keeping its birth's time fixed, next we integrate the birth's time over the segment $[0,1)$, this gives one. Then we come back to the initial indexation of the hoppings.

$$|\Phi^\beta[\tau(\vec{\mathscr{A}}^{\mathscr{P}})]|\leqslant R(\tau(\vec{\mathscr{A}}^{\mathscr{P}}))[r(\tilde{U})-|r(\tilde{\mu}_e)|]\prod_{k=o}^{k=n}|\hat{T}^{[1]}_{A_k}| \tag{C.17}$$

To get the final estimate, we bound the coefficient $R(\tau(\vec{\mathscr{A}}^{\mathscr{P}}))$, which is a geometric coefficient, which depends only of the intersection properties of the cluster.[21] Then $R(\tau(\vec{\mathscr{A}}^{\mathscr{P}}))$ is the same for all the clusters, which have the same tree $\tau(\vec{\mathscr{A}}^{\mathscr{P}})$. So we use the Cauchy formula for analytic functions, to get an upper bound on $R(\tau(\vec{\mathscr{A}}^{\mathscr{P}}))$.

$$|\phi^{\beta}[\tau(\vec{\mathscr{A}}^{\mathscr{P}})]| \leqslant \sup_{\{T_A{}^i\}_{i\in\{o,\ldots,n\}}} |Z_{\{\bigcup_{i=1}^{i=p}\mathscr{F}_{[o,1)}(\vec{\mathscr{A}}^i)\}}| \times \frac{|\hat{T}_{A_o}^{[1]}| \prod_{k=1}^{k=n} |\hat{T}_{A_k}^{[1]}|}{|\bar{T}_{A_o}| \prod_{k=1}^{k=n} |\bar{T}_{A_k}|} \tag{C.18}$$

The "sup" is taken over the T_{A^i}, which are the rescaled hoppings satisfying an S.I. condition, such upper bounds exist because the hoppings decay exponentially in the tree distance. To bound the partition function, we use the quantum Dobrushin's inequality contained in the Proposition B.2.

$$\sup_{\{T_{A^i}\}} |Z_{\{\bigcup_{i=1}^{i=p}\mathscr{F}_{[o,1)}(\vec{\mathscr{A}}_i)\}}| \leqslant \left[\sum_{i=o}^{i=n} |\bar{T}_{A_o^i}| \prod |\bar{T}_{A_o^i}|\right] \tag{C.19}$$

We define the rescaled hopping: $|\hat{T}_{A_k}^{[2]}| =: \frac{|\hat{T}_{A_k}^{[1]}|}{|\bar{T}_{A_k}|}$ to get the estimate:

$$|\phi^{\beta}[\tau(\vec{\mathscr{A}}^{\mathscr{P}})]| \leqslant [r(\tilde{U}) - |r(\tilde{\mu}_e)|] \prod_{k=o}^{k=n} |\hat{T}_{A_k}^{[2]}| \tag{C.20}$$

Note. The upper bound obtained for $|\phi^{\beta}[\tau.\vec{\mathscr{A}}^{\mathscr{P}}]|$ depends ONLY of the rescaled hoppings. No matter of the tree $\tau(\vec{\mathscr{A}}^{\mathscr{P}}) \in \mathscr{T}(\vec{\mathscr{A}}^{\mathscr{P}})$, of the constituent loops of the cluster, provided that the clusters are projected onto the same linked graph $\mathscr{A}^{\mathscr{P}}$.

3.2.6. Step II. The Cardinality $\mathscr{E}(\vec{\mathscr{A}}^{\mathscr{P}})$ of the Spanning Trees of $\mathscr{T}(\vec{\mathscr{A}}^{\mathscr{P}})$

The cardinality of the spanning trees, which are built on the graph $C^*[T, \vec{\mathscr{A}}^{\mathscr{P}}]$, is estimated by using a local estimate. The $2k_x$ edges incident at x are partitioned into r subsets of identical edges, each one has a multiplicity η_i^x. Then $\mathscr{E}(\vec{\mathscr{A}}^{\mathscr{P}})$ is bounded by the product over the vertices of $\vec{\mathscr{A}}^{\mathscr{P}}$ of the sum of the different partitions of even incident edges at the vertices x of the graph $\vec{\mathscr{A}}^{\mathscr{P}}$.

- The number of decompositions into k_x pairs of edges built from $2k_x$ edges, is bounded by $\frac{2k_x!}{2^{k_x}} \times \frac{1}{\eta_1^x! \times \cdots \times \eta_r^x!}$.
- The number of partitions of a given set of k_x pairs of edges into partitions of subsets is bounded by 2^{k_x}. Collecting these bounds, we get:

$$\mathscr{E}(\vec{\mathscr{A}}^{\mathscr{P}}) \leqslant \prod_{\{x\in\vec{\mathscr{A}}^{\mathscr{P}}\}} \frac{(2k_x)!}{\eta_1^x! \times \cdots \times \eta_r^x!} = \prod_{\{x\in\vec{\mathscr{A}}^{\mathscr{P}}\}} \frac{[\sum_{i=1}^{i=r} \eta_i^x]!}{\eta_1^x! \times \cdots \times \eta_r^x!} \tag{C.21}$$

3.2.7. Step III. The Cardinality $\mathscr{E}(\mathscr{A}^{\mathscr{P}})$ of the Constituent Complete Families of Loops

The set of clusters, which are projected on the same unoriented graph $\mathscr{A}^{\mathscr{P}}$, can be built from different complete families of loops, two non winding intersecting loops of $\mathscr{L}_2^o$, and a loop of $\mathscr{L}_4^o$ can have the same projection and the same jumps. We bound the cardinality of the complete families of loops, which have the same family of jumps, the same projection onto $\mathscr{A}$. Again we look for a local bound at each vertex $x \in \mathscr{A}^{\mathscr{P}}$, next they will be factorized the estimates obtained for all the vertices of $\mathscr{A}^{\mathscr{P}}$. Let us start from the set of the $2k_x$ edges which are incident at a vertex x, we consider the different partitions of these $2k_x$ edges. The set of the different complete families of the loops contained in the cluster is included into the set of the even different partitions of the $2k_x$ of the edges containing x.

$$\mathscr{E}(\mathscr{A}^{\mathscr{P}}) \leqslant \prod_{\{x \in \vec{\mathscr{A}}^{\mathscr{P}}\}} \frac{(2k_x)!}{\eta_1^x! \times \cdots \times \eta_r^x!} \tag{C.22}$$

The cardinality of the pairings of $\mathscr{A}^{\mathscr{P}}$ of the set $\mathscr{A}$ is given by: $\prod_j \frac{|A_j|}{2}!$. Finally we insert the estimates (C.21) and (C.22) to get (C.23):

$$\begin{aligned}
|\Psi^{\beta}[\Sigma_{\mathscr{A}}]| &\leqslant [r(\widetilde{U}) - |r(\widetilde{\mu}_e)|] \left\{ \prod_{i=o}^{i=n} \frac{|A_i|}{2}! \times |\hat{T}_{A_i}^{[2]}| \right\} \times \mathscr{E}(\vec{\mathscr{A}}^{\mathscr{P}}) \times \mathscr{E}(\mathscr{A}) \\
&\leqslant [r(\widetilde{U}) - |r(\widetilde{\mu}_e)|] \times \left\{ \prod_{i=1}^{i=n} \frac{|A_i|}{2}! \times |\hat{T}_{A_i}^{[2]}| \right\} \left\{ \prod_{\{x \in \vec{G}\}} \frac{(2k_x)!}{\eta_1^x! \times \cdots \times \eta_r^x!} \right\}^2 \\
&\leqslant [r(\widetilde{U}) - |r(\widetilde{\mu}_e)|] \left[\prod_{i=o}^{i=n} \frac{|A_i|}{2}! \times |\hat{T}_{A^i}^{[3]}| \right]
\end{aligned} \tag{C.23}$$

We used again, at each vertex of the set $\mathscr{A}_x$, the formula (C.12) of the combinatorial lemma to remove the factorials appearing at every incoming edge by constants, we use the following fact: each edge of the graph contains two vertices, and then is counted at each vertex. The constants are absorbed in the new resealed hoppings: $|\hat{T}_{A_i^{\{o,j\}}}^{[3]}| =: \bar{\Phi}(H) \times |\hat{T}_{A_i^{\{o,j\}}}^{[2]}| \times |\bar{T}_{A_i^{\{o,j\}}}|^{-4}$. Notice that our bounds were obtained by considering the limit $S \to \infty$, which are upper bounds for the corresponding expressions for finite S. Next we use the quantum Dobrushin's inequality. The effective potentials can be defined as the multiple derivatives of the log of the conditional partition functions. The existence of the limit $S \to \infty$ of the conditional partition functions as analytic functions proved in the Proposition B.1.2

implies the existence of the limits $S \to \infty$ of the correlation functions as analytic functions. We notice that the contributions of the winding loops without jumps is given by: $\beta^{-1} \log[1+\exp(-\beta(U-\sigma_v \mu^e)]$.

3.2.8. (b) The Low Temperature Estimate

To get the low temperature estimate, we have to sum over all the truncated functions of the clusters containing one point, we modify the above estimate as follows: we go back to the expression of $S[\Lambda^o \curlywedge \lambda^1]$, in which we factorize the s.p. of the winding loop from the other terms. Then we sum independently the two factors as follows:

- first we sum over all the s.p. of a winding loops containing one point using the estimate (b) of the Proposition A.2.
- secondly we bound the other factors by using the estimate (C.5) of the Proposition C.2.2.

$$
\begin{aligned}
|R_p^{\{\beta, w\}}[\Sigma_V \mid \Sigma_{\bar{V}}]| &\leqslant \beta^{-1} \exp -\beta(r(U)-|r(\mu_e)|-\|T\|-\|T\| O\left(\frac{\|T\|}{r(U)}\right\}\right) \\
&\quad \times\left[1+\frac{\|T\|}{r(\widetilde{U})-|r(\widetilde{\mu}_e)|-\|T\|}\right] \\
&\leqslant \frac{\|T\|^{p+1}}{[r(\widetilde{U})-|r(\widetilde{\mu}_e)|]^p}
\end{aligned}
\tag{C.24}
$$

APPENDIX D. THE EFFECTIVE HAMILTONIAN: PROOF OF THE THEOREM

D.1. The General Case: The *p* Order Decomposition of the Hamiltonian $\mathscr{H}_\beta$

We can write the conditional partition functions as:

$$
Z\{H; \Sigma_V \mid \Sigma_{\bar{V}}]\} = \exp -\beta\left\{\mathscr{H}_\beta^o[\Sigma_V \mid \Sigma_{\bar{V}}) + \sum_{\{\Sigma_{\mathscr{A}} \in \Sigma_V \cup \Sigma_{\bar{V}}\}} \Psi^\beta[\Sigma_{\mathscr{A}}]\right\} \tag{D.1}
$$

It is crucial to see that the potentials constructed through the C.E. are local: they depend ONLY of the local ion's configuration restricted to their

support. The hamiltonian $\mathbf{H} \in \mathbf{F}$ has an effective hamiltonian $\mathscr{H}_\beta$ uniquely defined by its value on each ion's configuration $(\Sigma_V \mid \Sigma_{\bar{V}})$.

$$\mathscr{H}_\beta[\Sigma_V \mid \Sigma_{\bar{V}}] = \mathscr{H}^o_\beta[\Sigma_V \mid \Sigma_{\bar{V}}] + \cdots + \mathscr{H}^p_\beta[\Sigma_V \mid \Sigma_{\bar{V}}] + \sum_{\{\Sigma_{\mathscr{A}} \in \Sigma_{\bar{V}} \mid |\mathscr{A}| > p\}} \Psi^\beta[\Sigma_{\mathscr{A}}] \tag{D.2}$$

The truncated hamiltonians axe computed explicitly up to the order p (i.e., the potentials built from the truncated functions of the clusters containing p hoppings). The tail potentials, decay exponentially in the tree distance (Proposition C.2.2.).

D.2. The LT *p* Order Decomposition of the Hamiltonian $\mathscr{H}_\beta$

$$\mathscr{H}_\beta(V) = \mathscr{H}^o_\infty(V) + \cdots + \mathscr{H}^p_\infty(V) + \mathscr{H}^{\leqslant p}_{\{w,\beta\}}(V) + \mathscr{H}^{> p}_\beta(V) \tag{D.3}$$

We partition the truncated functions into two parts.

(1) The truncated functions, which do not involve winding loops. Notice that this corresponds to take $\beta = \infty$ in the effective potentials $\mathscr{H}^p_\beta[\Sigma_V \mid \Sigma_{\bar{V}}]$, because the effective potentials, which are built from the truncated functions of the clusters, which contain a winding loop, go to zero, when $\beta \to \infty$.

(2) The truncated functions, which do involve winding loops.

$$\mathscr{H}^{\leqslant p}_{\{w,\beta\}}(V) := \mathscr{H}^o_{\{w,\beta\}}(V) + \cdots + \mathscr{H}^p_{\{w,\beta\}}(V)$$

The estimate (b) of the Proposition C.2.2. provides an upper bound on the potentials of $\mathscr{H}^{\leqslant p}_{\{w,\beta\}}(V)$ at low temperature.

APPENDIX E. THE EFFECTIVE HAMILTONIAN OF $H \in F$; $J_{X,Y} \neq 0$

The definitions contained in the previous appendixes, are transposed in the case of the interacting linked loops just by adding the subscript I. The arguments of the functions, defined in the previous appendixes, include the coupling constants $J_{\{X,Y\}}$. The basic tool needed to construct the effective hamiltonian $\mathscr{H}_\beta$ of an hamiltonian $H \in F$ is again the existence of a convergent C.E. extended to the interacting linked loops. The polymers are now the interacting loops, i.e., the connected sets of loops, which are at distance smaller than l. Two polymers are compatible if the distance between every pairs of loops belonging to the different polymers is larger

than the range of the classical hamiltonian l. Notice that the exact knowledge of the s.d. of the interacting loops is required only when we compute explicitly the first orders of the effective hamiltonian. For the higher order terms we just need an upper bound on each s.d., to be able to prove the existence of a convergent C.E..

E.1. The Estimates

We generalize the estimates of the Appendix A to the interacting loops $\hat{\lambda}^I$. We first bound the absolute values of the s.d. of $\hat{\lambda}^I$ by the products of the s.d. of the constituent loops of $\hat{\lambda}^I$, in which we replace the classical energy of the vertical up arrows and of the down arrows of the loops by the upper bound of the energy $\hat{U} - 2\sum_{\{X, Y \in V \times V \mid o \in X\}} |J_{\{X,Y\}}| := U$. Now the interaction between two loops is revealed by the geometric constraint: the distance between the two loops is smaller than l. Next we derive the two main estimates, which extend those of the Appendix A.

Proposition E.1.1. The hamiltonian $\mathbf{H} \in \mathbf{F}$ satisfies an S.I. condition (0.11).

$$|r(\tilde{\mu}_e)| < r(\tilde{U}) - l^{2\nu}\|T\| - \exp -\beta\{r(\tilde{U}) - |r(\tilde{\mu}_e)| - l^{2\nu}\|T\|\} + h.o. \quad \text{(E.1)}$$

(A) Then the following estimates hold for every temperature:

$$\text{(i)} \quad |\Phi_I^p[\mathscr{F}_{\{0,1\}}(\vec{\mathscr{A}})]| \leqslant \frac{p!}{[r(\tilde{U}) - |r(\tilde{\mu}_e)|]^p} \left\{ \prod_{i=1}^{i=n} l^{2\nu} \cdot 2^{\frac{A_i}{2}} |t_{A_i}| \right\} \times \left\{ \frac{1}{r(\tilde{U}) - |r(\tilde{\mu}_e)|} \right\}^{n-1} \quad \text{(E.2)}$$

$$\text{(iv)} \quad |\Phi_I^p[U, T, \mu_e]| \leqslant \beta^{-1} \exp(-\beta[(U) - |r(\mu^e)|) + \left\{ \frac{p!}{[r(\tilde{U}) - r(\tilde{\mu}^e)]^p} \times \frac{l^{4\nu}\|T\|^2}{r(\tilde{U}) - |r(\tilde{\mu}_e)| - l^{2\nu}\|T\|} \right\}$$

(a) There exists two positive constants C_p and A_p such that, if $\beta \frac{l^{2\nu}\|T\|^{p+1}}{[r(\tilde{U}) - \|T\|]^p} > A_p$, we have:

$$|\Phi^I[U, T, \mu_e]| \leqslant C_p \frac{[l^{2\nu}\|T\|]^{p+1}}{[r(\tilde{U}) - \|T\|]^p} \quad \text{(E.3)}$$

Proof. The geometrical structure of the clusters of non interacting loops defined above and the structure of the interacting loops are similar, if

we make the following correspondance: the notion of intersection between two loops $\hat{\lambda}$ and $\hat{\lambda}'$ of a cluster is replaced by the notion of interaction between the loops $\hat{\lambda}$ and $\hat{\lambda}'$, i.e., that the two loops are such that $o < d(\hat{\lambda}; \hat{\lambda}') < l$. Then the generalisation of the estimates of the Appendix A to the interacting loops is obtained from the estimates performed in the Appendix C for the sum of the s.d. of the complete families of clusters. We compare the situation of a cluster and of an interacting loop which constituent loops are the same. Let us consider a cluster containing two intersecting loops $\hat{\lambda}$ and $\hat{\lambda}'$. In the computations of the Appendix C, the loop $\hat{\lambda}$ is fixed, meanwhile the births' times of $\hat{\lambda}'$ is integrated. Let us go to the case of the interacting loops $\hat{\lambda}$ and $\hat{\lambda}'$, the loop $\hat{\lambda}$ is integrated, then the birth's time of $\hat{\lambda}'$ is integrated over a cylinder which basis is a v hypercube of basis length l, and which height is the time. So the estimates for the sums of s.d. of interacting loops of the above proposition are deduced from the estimates of the Proposition C.2, if we insert a factor l^{2v} for each pair of loops which interact. As there is, at least, one hopping per interaction, we get the upper bounds contained in the proposition, if we insert a factor l^{2v} in each rescaled hopping, in the estimates obtained in the proof of the Proposition C.2. This defines the new S.I. condition (0.11), which is we needed for the proposition. ▌

E.2. The Dobrushin's Inequalities

The proofs are similar to the case of non interacting loops. The incompatibility relation between two interacting loops is that at least two loops, one for each interacting linked loop, intersect, then we use the estimates contained in the Proposition D.1.1. to get the new S.I. condition:

$$|r(\tilde{\mu}^e)| < r(\tilde{U}) - l^{2v} \|T\| - \exp - \beta\{r(\tilde{U}) - |r(\tilde{\mu}_e)| - l^{2v} \cdot \|T\|\} \qquad \text{(E.4)}$$

E.3. The Exponential Decay of the Potentials

Proposition E.3.1. The hamiltonian **H** belonging to **F** fulfill an S.I. condition.

(a) The potentials contained in the effective hamiltonian $\mathscr{H}_\beta$ decay exponentially.

$$|\Psi^\beta[\Sigma_{\mathscr{A}}]| \leqslant [r(\tilde{U}) - |r(\tilde{\mu}_e)|] \times \prod_{i=1}^{i=n} 2^{\frac{A_j}{2}} \times \frac{A_j}{2}! \times |l^{2v} \hat{T}_{A^i}^{[3]}| \leqslant \exp - K.T(\mathscr{A}) \qquad \text{(E.5)}$$

(b) Moreother for $\beta \frac{[l^{2\nu}\|T\|]^{p+1}}{[r(\widetilde{U})-|r(\widetilde{\mu}_e)|]^p} > A_p$, and for $|\mathscr{A}| < p$, we get:

$$|R_p^{\{\beta, w\}}[\Sigma_V \mid \Sigma_{\bar{V}}]| \leqslant \frac{[l^{2\nu}\|T\|]^{p+1}}{[r(\widetilde{U})-|r(\widetilde{\mu}_e)|-l^{2\nu}\|T\|]^p} \tag{E.6}$$

Proof. To prove the exponential decay of the effective potential, we use exactly the same route as for the non interacting case. We have to take into account three facts.

- We extend the C.E. performed for non interacting loops to the interacting loops.
- When integrating over the s.p. of the interacting loops, we use the estimates of the proposition D.I.1., which requires that a new S.I. condition is satisfied.
- When two loops contained in two interactings loops intersect, the times' integrals over their birth's time is the same as for the case of non interacting loops.

When an S.I. condition holds, the potentials of the effective hamiltonian decay exponentially, if we insert in the computations of the Appendix C a factor $l^{2\nu}$ in front of every rescaled hopping. ▌

6. CONCLUSIONS

The study of a class of models of the quantum statistical mechanics was converted into a solvable problem of the classical statistical mechanics. In a companion paper,[5] we apply our approach to the Falicov–Kimball model. Under the S.I. condition, the existence of a cascade of new phase transitions with higher periodicities is shown to exist. We point out that T. Kennedy and E. Lieb have proved the existence of a phase transition even when the weaker W.I. condition holds for $\mu^e = \mu^i = U$, and arbitrary $\frac{t}{U} > 0$. So that we expect that a part of our result should be true in this case, however we do not expect that phases of period larger than two should exist. Fermions with spin are more difficult to treat, but our method can be extended to prove the existence of a Neel phase transition at low temperature for the assymetric Hubbard model (different hoppings) by using a contour representation in the space time. The loop expansion can be done for bosonic systems. We mention the Holstein model, which takes into account the vibrations of the ions (polarons).[25, 26, 28] Our method can be generalized to the static Holstein model considered in ref. 27. After subsequent generalisations, one should be able to prove the Peierls instability in the Holstein model.

Our approach makes possible to study the problem some quantum interfaces using again the methods of the classical statistical mechanics. The structure which appears for the phase transitions appears for the quantum interfaces of the FK model. The quantum fluctuations select one dominant quantum interface. The 100 interface of the FK model (the interface orthogonal to the 100 vector) is rigid at low temperature due to the first order quantum fluctuations. The 111 interface is infinitely degenerate for the first order effective hamiltonian. The third order quantum fluctuations together with the Fermi statistic are responsible of the rigidity of the 111 interface at low temperature.[29] We point out the papers of Datta, Fernandez, and Frohlich,[30a, 30b] which show the existence of quantum phase transitions.

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